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Estimates for Stress Derivatives and Error in Interior Equations for Shells of Variable Thickness with Applied Forces

Neil Berger

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ABSTRACT

Estimates for the derivatives of the stresses in thin shells and plates have been derived by Fritz John in the case where there are no body forces, no surface forces, and the medium is of constant thickness.

The procedure used has been modified to derive similar estimates in the case of a continuum of variable thickness acted upon by body and surface forces. It is assumed that the strains and external forces on the undeformed shell are bounded and small, and that the curvature and variation in thickness of the shell are not too large. As a result solely of these assumptions and the three-dimensional equations of non-linear elasticity rigorous estimates are obtained on the stresses and their derivatives in a region away from the lateral edges of the shell. These estimates depend only on the strain energy density function and the physical dimensions of the shell.

Starting from the mathematical three-dimensional equations of non-linear elasticity, one can derive, as a consequence of these estimates, a set of two-dimensional approximate differential equations for shells with precise estimates on the error. These new equations take into account the variable thickness of the shell and loading on the surface and they contain as a consequence those of v. Karman for plates of constant thickness with normal loads.

Introduction

We consider a perfectly elastic homogeneous and isotopic shell in equilibrium with applied surface, edge, and body forces. The faces of the shell in the unstrained state shall be of variable distance from the middle surface. Under the action of the applied forces, the shell deforms to a new, strained position.

We apply techniques derived from the theory of elliptic partial differential equations to obtain a priori estimates for stress components and their derivatives which can be used to formulate equations which describe the deformed state of the shell.

These estimates have been established by Fritz John [3] for shells of constant thickness with no applied forces. The techniques used are extended and modified to establish similar estimates for an elastic layer of variable thickness with applied forces. No ad hoc assumptions about the relative orders of magnitude of the transverse and longitudinal stresses and their derivatives ("Kirchoff hypotheses") are employed.

The equations formulated reflect the variations in thickness of the shell and the presence of body forces. These equations are similar to the classical equations due to v. Karman which have been used to study plates of variable thickness, plates on elastic foundations, and buckling of plates. In the case where the shell is of constant thickness and there are no applied forces, the formulated equations are similar to those derived by Fritz John.

A knowledge of the geometry of the shell in its unstrained state and some elastic description of the shell--stress strain law--and bounds on the applied forces are all that are needed to determine these estimates.

If the thickness of the shell is small and does not vary too wildly, the curvature of the middle surface is not too large, and the strains are small everywhere in the shell, then certain estimates on the stresses and their derivatives are valid.

To make our assumptions rigorous, we define:

h = undeformed variable distance to the center plane of the shell from the surface;
R = "generalized" radius of curvature, a constant;
D = distance from the point P under consideration in the shell to the lateral boundary ("edge") of the shell;
H = maximum of h over the region of consideration;
H^* = minimum of h over the region of consideration;
ε = upper bound for the strain everywhere in the shell.

The quantity

$$\theta = \max\left[\frac{2H}{D}, \frac{2H}{\sqrt{2H^*R}}, \frac{H}{H^*}\cdot\sqrt{\varepsilon}\right]$$

can be used as a common measure for the "thinness," "shallowness" and "variation of thickness" of the shell, and of the linearity of the generalized Hooke's law at the point P in the undeformed middle surface of the shell, which is the center of a disk of radius D.

If θ is small, a two-dimensional approximate description of the deformed state of the shell becomes feasible. The quantity λ defined by

$$\lambda = \frac{2H}{\theta} = \min\left(\frac{D}{3}, \sqrt{2H^{*}R}, \frac{2H^{*}}{\sqrt{\varepsilon}}\right)$$

provides a lower bound for the "wave length" of the stresses and strains in the longitudinal directions.[1]

In general it will be shown that the n-order longitudinal derivatives of the strain components are bounded by $K_n\varepsilon/\lambda^n$ when $\theta < \theta_0$, and the constants θ_0, K_1, K_2, $\cdots$ depend on the elastic properties of the material alone. In addition, conditions will be imposed on the order of magnitude of the applied forces in order that the stresses internal to the shell be of similar order. If, as assumed, the maximum stress is of order E (where E is Young's modulus), it can be shown that the transverse shear stress are of order $E\varepsilon\frac{h}{\lambda}$ at each point in the interior of the shell, and that the longitudinal stress varies linearly within an order of error of magnitude $E\varepsilon\frac{h^2}{\lambda^2}$ along a normal fiber of the shell. Here the expression "order of error" indicates that any constants tacitly involved in the estimate can at most depend on the material (stress--strain description chosen).

Once these estimates have been obtained, approximate two-dimensional equations ("interior shell equations") involving surface and body forces can be derived from the exact equations of a three-dimensional elastic media. This is accomplished by grouping terms in the equations according to powers of h/λ appearing in the concrete estimates, and hence the errors of approximation will be known.

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[1] By wavelength of a function f, we mean a quantity proportional to $\max|f|/|f'|$.

The technique used is that of deriving bounds on the Hilbert norms of the derivatives of the stresses from energy identities. These bounds can be converted into pointwise estimates by means of a Sobolev type lemma. Since the equations involved are non-linear, it is clear that estimates for the norms must involve this kind of technique.

In proposing a system of shell equations we choose as the unknown functions the three-longitudinal components of stress in the middle surface and the three coefficients of the second fundamental form of the deformed middle surface. The three-dimensional theory of elasticity furnishes six non-trivial approximate equations for these six quantities.

If in addition the shell is very shallow, then these six approximate equations can be further reduced to two equations for two unknown functions. One of these is analogous to the Airy stress function, and the other is related to an approximate integral of the second fundamental form of the deformed middle surface of the shell. In the case where the shell is of constant thickness and has no initial curvature, this system of equations agrees with that ascribed to v. Karman [4] and Föppl. In all the systems of equations mentioned above, terms are included which arise from the variable thickness of the shell, and the body and surface forces. In particular, a term due to normal equal and opposite loading of the shell appears.

Chapter 1 - Summary of the Equations of a Homogeneous Isotropic Perfectly Elastic Solid in Equilibrium with Body and Surface Forces[2]

1.1 Coordinate System

We use Lagrange coordinates $U = (U^1, U^2, U^3)$ to distinguish the individual particles of the solid. The solid is considered in two states--"unstrained" and "strained." Let the Cartesian coordinates of the same particle be $X^i = X^i(U)$ for the unstrained state, and $x^i = x^i(U)$ for the strained state. By $f_{,i}$ we denote the partial derivative of the function f with respect to the variable U^i. Line elements in the unstrained and strained states respectively are

$$dS^2 = dX^i dX^i = G_{ik} dU^i dU^k \quad \text{and}$$
$$ds^2 = dx^i dx^i = g_{ik} dU^i dU^k, \quad \text{where}$$
$$G_{ik} = X^j_{,i} X^j_{,k} \quad \text{and} \quad g_{ik} = x^j_{,i} x^j_{,k}$$

are the unstrained and strained metric tensors. Although tensor operators can be performed in either metric, all shifting of indices will be performed with the unstrained metric tensor G_{ik}, which in principle is known. In addition the dS^2 metric can be used to express covariant differentiation with respect to the variables U^i in the ds^2 metric.

The Christoffel symbol for the dS^2 metric is

$$(1) \qquad \Gamma^i_{jk} = \tfrac{1}{2} G^{ik}(G_{rj,k} + G_{kr,j} - G_{jk,r})$$

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[2] See John [3], Truesdell [7].

and for the ds^2 metric

$$\bar{\Gamma}^i_{jk} = \frac{1}{2}\bar{g}^{ir}(g_{rj,k} + g_{rk,j} - g_{jk,r}) = \Gamma^i_{jk} + c^i_{jk}$$

where c^i_{jk} is the tensor:

$$(2)^3 \qquad c^i_{jk} = \frac{1}{2}\bar{g}^{ir}(g_{rj;k} + g_{rk;j} - g_{kj;r}),$$

";k" denotes covariant differentiation with respect to U^k in the dS^2 metric, and $\bar{g}^{ir}$ and g^{ir} are defined by

$$(2a) \qquad \bar{g}^{ir}g_{rk} = \delta^i_k, \quad g^{ir} = G^{im}G^{rs}g_{ms}.$$

The conditions of compatibility reflect the fact that ds^2 and dS^2 are Euclidian line elements. Expressing the vanishing of the Riemannian curvature tensor for the ds^2 metric in terms of the dS^2 metric, gives six independent conditions:

$$(3) \quad 0 = \frac{1}{2}(g_{hk;ij}+g_{ij;hk}-g_{hj;ik}-g_{ik;hj})+g_{ab}(c^a_{ij}c^b_{hk}-c^b_{hj}c^a_{ik}).$$

The principal strains are the eigenvalues of the strain matrix defined by

$$(4) \qquad \varepsilon^i_k = G^{ir}\varepsilon_{rk} = G^{ir}\,\frac{1}{2}(g_{rk}-G_{rk}) = \frac{1}{2}(g^i_k-\delta^i_k).$$

1.2 Strain Energy Density Function W, Stresses, and Equations

For a homogeneous isotropic perfectly elastic material, the strain energy density per unit unstrained volume is a

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[3] Numbers such as 3.2 will refer to Chapter 3, item 2.

symmetric function of the principal strains. We require that W agree with the classical strain energy density function for small strains.

The contravariant components of the stress tensor t^{mk} in the U^i coordinate system are defined in such a way that the force acting on an element of area ds with direction cosines ξ_r in the strained state has Cartesian x-components:

$$(5) \qquad t^{mk}x^i{}_{,m}x^r{}_{,k}\xi_r ds, \quad \text{for} \quad i = 1,2,3.$$

The stress-strain relations are given by

$$(6) \quad t^m_i = G_{ik}t^{mk} = \left(\frac{G}{g}\right)^{1/2}\frac{\partial W}{\partial \varepsilon_{mk}} = (G/g)^{1/2}\left(\frac{\partial W}{\partial s_1}\delta^m_i + 2\frac{\partial W}{\partial s_2}\varepsilon^m_i + 3\frac{\partial W}{\partial s_3}\varepsilon^m_j\varepsilon^j_i\right),$$

with $(g/G) = (\det g_{ij}/\det G_{ij}) = 1+2s_1+2(s_1^2-s_2)+4(2s_3-3s_1s_2+4s_1^3)$,
$S_1 = \varepsilon^i_i$, $S_2 = \varepsilon^i_k\varepsilon^k_i$, $S_3 = \varepsilon^i_j\varepsilon^j_k\varepsilon^k_i$.

The equations of equilibrium take the form

$$(7) \quad -B^m = t^{mk}{}_{;;k} = t^{mk}{}_{;k} + t^{ik}c^m_{ik} + t^{mi}c^k_{ik} = t^{km}{}_{,k} + t^{ik}\overline{\Gamma}^m_{ik} + t^{mi}\overline{\Gamma}^m_{ik}.$$

where ";;k" denotes the covariant derivative with respect to U^k formed in the ds^2 metric, and B^m is the U^m-component of the body force per unit deformed volume (normalized to agree with the unit of force chosen for the stresses).

1.3 Differential Equations of Elasticity in Pseudo-classical Form

The tensor or invariant form of the differential equations are inconvenient to use in estimating the derivatives of the stresses. Instead we regard the equations as between vectors. An object such as $t = (t^{ik})$, with any number of subscripts or superscripts, can be regarded as a vector--a point in a normed linear space of dimension three times the number of indices The norm of a vector shall be the square root of the sum of the squares of its components. For example, if $w = (w^{ik}_{rs})$, then the norm of w is $|w| = \sqrt{w^{ik}_{rs} w^{ik}_{rs}}$.

Vectors can be multiplied by scalars: $\lambda w = (\lambda w^{ik}_{rs})$. Vectors of the same type can be added: if $u = (u^{ik}_{rs})$ and $v = (v^{ik}_{rs})$ then $u + v = (u^{ik}_{rs} + v^{ik}_{rs})$. Any two vectors can be multiplied: if $w = (w^{ik}_{rs})$ and $v = (v^{ab}_{c})$ then $wv = (w^{ik}_{rs} v^{ab}_{c})$, $v^2 = (v^{ab}_{c} v^{de}_{f})$ and $w^2 = (w^{ik}_{rs} w^{\ell m}_{np})$. The gradient of a vector w with respect to the Lagrange coordinates U^i is again a vector, denoted by w': if $w = w^{ik}_{rs}$, then $w' = w^{ik}_{rs,t}$. We also have for any two vectors u and v: $|uv| = |u| \cdot |v|$.

In many cases, only the general form of an expression will be of interest. We use the following notation to display this. If u, v, w, p, q are vectors then $F(p,q)[u+v+w]$ will denote a vector valued function each of whose components is a linear form in u, v, and w, with coefficients that are functions of p and q, defined and differentiable as often as needed for sufficiently small p and q. F may, and in general will, stand for a different expression in each equation.

The assumptions on W indicate that its form is for $-1 < \nu < \frac{1}{2}$

(8) $$W = \frac{E}{2(1+\nu)(1-2\nu)}[\nu S_1^2+(1-2\nu)S_2] + F(S_1,S_2,S_3)[S_1^3+S_1S_2+S_3].$$

Adjusting the units of force normalizes Young's modulus E, leaving Poisson's ratio as the only classical elastic constant.

Introducing $t = (t^i_k)$, and $e = (\varepsilon^i_k)$, we have by (1.8), (1.6),

(9) $$t^m_i = \frac{\nu}{(1-2\nu)(1+\nu)} {}^j_j\delta^m_i + \frac{1}{1+\nu}\varepsilon^m_i + F(e)e^2.$$

Solving for e in terms of t we obtain a stress-strain relationship of the form:

(10) $$\varepsilon^m_i = (1+\nu)t^m_i - \nu t^j_j\delta^m_i + F(t)t^2.$$

We introduce the vector $\eta = (\eta^i_k)$ by the highly non-invariant equation

(11) $$G_{ik} = \delta^i_k + \eta^i_k.$$

Then,

(12) $$G^{ik} = \delta^i_k - \eta^i_k + F(\eta,t)\eta t, \quad \Gamma^i_{kj} = F(\eta)\eta',$$

(13) $$t^{ik} = t^i_k + F(\eta,t)\eta t = t_{ik} + F(\eta,t)\eta t,$$

and from (1.4) and (1.10) we have

(14) $$g_{ik} = G_{ik}+2\varepsilon_{ik} = G_{ik}+2(1+\nu)t_{ik}-2\nu t_{jj}\delta^i_k+F(\eta,t)[t^2+\eta t],$$

and from (1.2), (1.12), (1.14)

(15) $$c^i_{kr} = F(\eta,t)[t'+\eta' t].$$

The equations of equilibrium (1.7) become using (1.15), (1.12)

(16) $$t^{im}{}_{,m} + B^i = F(\eta,t)[tt'+\eta' t+\eta t'] = P^i.$$

The conditions of compatibility (1.7) yield, upon substitution of (1.11), setting $i = j$ and use of (1.13), (1.16):

$$(17) \quad t^{hk}{}_{,rr} + \frac{1}{1+\nu} t^{rr}{}_{,hk} + \frac{\nu}{1-\nu}(B^{j}{}_{,j})\delta_{hk} + B^{h}{}_{,k} + B^{k}{}_{,h} = Q^{hk}$$

$$= F(\eta,t)[\eta t'' + (t')^2 + \eta'' t + (\eta')^2 t + \eta' t' + t t''],$$

which resembles the Beltrami-Michell compatibility equations. In each of these equations, the expression $F(\eta,t)$ stands for a different function uniquely determined and regular in a neighborhood of the origin which is dependent solely on the choice of the strain energy density function W. We remark that by (1.13), it makes no difference if $t = (t^{ik})$ or $t = (t_{ik})$ in the expressions for P^i and Q^{hk}

1.4 "$\mathcal{O}$" and "σ" Notation

The symbol "$\mathcal{O}$" will be used in two ways. First in the conventional sense, and second to indicate dependence only on W.

The equation

$$A = \mathcal{O}(B), \quad B \geq 0$$

means that there exists a constant K dependent at most on the choice of W such that $|A| \leq KB$.

The symbol "σ" will be used only in combination with "$\mathcal{O}$" The equation

$$A = \mathcal{O}(B) + \sigma(C), \quad B \geq 0 \text{ and } C \geq 0,$$

means that there exists a function $K(k)$ defined for all positive k and dependent only on W such that

$$|A| \leq K(k)B + kC \quad \text{for all} \quad k > 0.$$

For example,

$xy = \mathcal{O}(|x|^2) + \sigma(|y|^2)$ where a possible choice for $K(k) = \frac{1}{4k}$ independent of the choice of W.

Rules for the use of the symbols "$\mathcal{O}$" and "σ" easily proved are:

$$A = \mathcal{O}(B) + \sigma(|A|) \rightarrow A = \mathcal{O}(B),$$
$$A = \mathcal{O}(B) + \sigma(C+|A|) \rightarrow A = \mathcal{O}(B) + \sigma(C),$$
$$A = \mathcal{O}(B) + \sigma(C) \rightarrow A = \mathcal{O}(B+C),$$
$$A = \mathcal{O}(B) + \sigma(C), \quad A' = \mathcal{O}(B') + \sigma(C') \rightarrow A+A' = \mathcal{O}(B+B') + \sigma(C+C')$$
$$A = \mathcal{O}(B) + \sigma(C), \quad C = \mathcal{O}(D) + \sigma(E) \rightarrow A = \mathcal{O}(B) + \sigma(D+E),$$
$$A = \mathcal{O}(B) + \sigma(C), \quad B = \mathcal{O}(P) + \sigma(E) \rightarrow A = \mathcal{O}(D) + \sigma(C+E).$$

Chapter 2 - Description of the Shell

Greek letters $\alpha, \beta, \gamma, \cdots$ will be used for indices ranging over 1 and 2, while Roman letters $i, j, k, \cdots$ will be used for indices ranging over 1, 2, and 3.

In the unstrained state the boundary of the shell shall consist of two faces Σ_{h+}, and Σ_{h-} and a lateral surface ("edge") L. We assume that the functions h+ and h- are well enough behaved so that it is possible to define a unique midsurface Σ_0 of the shell that is equidistant from the faces Σ_{h-} and Σ_{h+}. We consider the edge L to be composed of normals to Σ_0.

We choose the Lagrange coordinates U^i by the condition that all particles on the same normal to Σ_0 in the unstrained state have the same U^α coordinates, and distance U^3 from the midsurface Σ_0.

The coefficients $E_{\alpha\beta}$ and $L_{\alpha\beta}$ of the first and second fundamental forms of the undeformed midsurface Σ_0 will determine Σ_0 to within a rigid motion. The metric tensor of the unstrained space line element can, as a result of our special assumptions on our "chosen" Lagrange coordinate system, be written as

$$(1) \qquad G_{ik} = (E_{\alpha\beta} - 2U^3 L_{\alpha\beta} + (U^3)^2 L_{\alpha\gamma} L^\gamma_\beta)\delta^\alpha_i \delta^\beta_k + \delta^3_i \delta^3_k .$$

Selecting any point P_0 of Σ_0 as the origin of a Cartesian coordinate system X^i with X^3 axis normal to Σ_0 at P_0, we assume that we can represent the surface Σ_0 by the equation $X^3 = f(X^1, X^2)$ in a neighborhood of P_0. By a rigid motion of the undeformed Cartesian coordinate system, we can arrange that $f(0,0) = f_{,\alpha}(0,0) = 0$. We make the further assumption that actually the entire surface Σ_0 can be represented by $X^3 = f(X^1,X^2)$ with X^α varying over some domain D^*.[4]

We define the thickness of the shell--the distance between Σ_{h+} and Σ_{h-} measured perpendicular to the midsurface Σ_0--as $2h(X^1,X^2)$. Half the maximum and minimum thickness will be defined as

$$H^* = \inf_{X^\alpha \text{ in } D^*} h(X_1,X_2), \quad H = \sup_{X^\alpha \text{ in } D^*} h(X_1,X_2).$$

We associate a "characteristic radius" R with the surfaces Σ_h, Σ_0, Σ_{-h} as follows: for any choice of the origin P_0 of Σ_0, X^α in D^* the k^{th}-order derivatives of f with respect to X^α satisfy everywhere in D^*.

$$|f^{(k)}| \leq R^{-1}(hR)^{1-k/2} \quad k = 2,3,4,\cdots. \tag{2a}$$

At some point in D^* such that $X^\alpha X^\alpha < 2H^*R$

$$\left|\frac{\partial}{\partial X^\alpha}(f-h)\right| < (h/R)^{1/2}. \tag{2b}$$

Conditions (2a) and (2b) amount to the fact that R is an upper bound for the radius of curvature of the midsurface and faces of the shell. and that at some point in the shell $\Sigma_{\pm h}$ and Σ_0 are approximately parallel.

Choosing U^α which coincide locally with X^α we have for the first and second fundamental forms of the undeformed shell in terms of $f(U^1,U^2)$

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[4] This is more than is needed. See John [3], p. 244, footnote 13.

$$(3)\qquad \begin{aligned} E_{\alpha\beta} &= \delta^{\alpha}_{\beta} + f_{,\alpha}f_{,\beta}, \\ L_{\alpha\beta} &= (1+f_{,\gamma}f_{,\gamma})^{-1/2}f_{,\alpha\beta}. \end{aligned}$$

On the faces of the shell we have in terms of ξ_i-- the direction cosines of the normal to deformed surface

$$0 = \xi_i dx^i = \xi_i(x^i{}_{,\alpha}dU^{\alpha} + x^i{}_{,3}h_{,\alpha}dU^{\alpha}).$$

Denoting the surface forces per unit deformed surface area acting in the strained state in the x coordinate system by S_x^{i+} on Σ_h and by S_x^{i-} on Σ_{-h} we have from (1.5) and (2.3)

$$\begin{aligned} S_x^{i+} &= (t^{m3+} - h_{,\alpha}t^{m\alpha+})x^{i+}{}_{,m}(x^r{}_{,3}\xi_r)^+, \\ S_x^{i-} &= (t^{m3-} + h_{,\alpha}t^{m\alpha-})x^{i-}{}_{,m}(x^r{}_{,3}\xi_r)^-, \end{aligned}$$

for $i = 1,2,3$.

Multiplying by $\frac{\partial U^j}{\partial x^i}$ and summing, we have for S^{j+}, S^{j-}, the surface forces per unit deformed surface area acting in the strained state, but specified in the Lagrange coordinate system

$$\begin{aligned} S^{j+} &= (t^{j3+} - h_{,\alpha}t^{j\alpha+})(x^r{}_{,3}\xi_r)^+, \\ S^{j-} &= (t^{j3-} + h_{,\alpha}t^{j\alpha-})(x^r{}_{,3}\xi_r)^-. \end{aligned}$$

Chapter 3 - Parameters Used in Making *A Priori* Estimates on the Derivatives of the Stresses

Let P_0 be any point on the midsurface Σ_0, and D the closest distance of P_0 from the lateral surface of the shell L. Let R denote the characteristic radius associated with the middle surface Σ_0. We have by our assumptions on h that $H^* \leq |h(U^1,U^2)| \leq H$, where the shell has thickness $2h$.

Consider strains, shell thicknesses, surface loads S^{i+} on the face $U^3 = h$ and S^{i-} on the face $U^3 = -h$ which are specified in our Lagrange coordinate system and body forces B^i, also specified in our Lagrange coordinate system. In the region $M = \{(U^1,U^2,U^3) \,|\, U^\alpha U^\alpha < D^2,\ |U^3| < h(U^1,U^2)\}$ we require that these quantities satisfy the following conditions given in terms of a constant θ_0, to be specified in the sequel:

$$(1)\quad \frac{2H}{D} < \theta_0,\ \left(\frac{H}{H^*}\right)^2 \frac{2H^*}{R} < \theta_0^2,\ \left(\frac{H}{H^*}\right)^2 \left[\text{spectrum}\ \ (\varepsilon^i_k)\right] < \theta_0^2,$$
$$\left(\frac{H}{H^*}\right)^2 |S^{i\pm}{}_{,\alpha_1 \cdots \alpha_m}| < \theta_0^2 H^{-m},\quad \left(\frac{H}{H^*}\right)^2 |B^i{}_{,k_1 \cdots k_{m-1}}| < \theta_0^2 H^{-m}$$

where $i,k = 1,2,3$ and $m = 0,1,\ldots,N$; N will be picked as high as needed in the sequel.

We define

$$(2)\qquad \theta = \max\left(\frac{2H}{D},\ \frac{2H}{\sqrt{2H^*R}},\ \frac{H}{H^*}\sqrt{\varepsilon}\right).$$

If H, H^*, R, D are fixed, then θ is a non-decreasing function of ε. Let ε be determined as the smallest positive number such that[5]

$$(3)\quad \operatorname{spec}(\varepsilon^{i}_{k}) \le \varepsilon,\ |S^{\alpha\pm}{}_{,\alpha_1\cdots\alpha_m}| \le (\theta/\theta_0)^{m+1}H^{-m}(h/H)\varepsilon,$$

$$|S^{3\pm}{}_{,\alpha_1\cdots\alpha_m}| \le (\theta/\theta_0)^{m}H^{-m}\varepsilon,$$

$$|(S^{3+}-S^{3-})_{,\alpha_1\cdots\alpha_m}| \le (\theta/\theta_0)^{m+2}H^{-m}(h/H)^2\varepsilon,$$

$$|B^{i}{}_{,k_1\cdots k_{m-1}}| \le \begin{cases} (\theta/\theta_0)^{m+1}H^{-m}\varepsilon & \text{if the number of } U^3 \text{ derivatives is even and } i \ne 3, \text{ or the number of } U^3 \text{ derivatives is odd and } i = 3, \\ (\theta/\theta_0)^{m+2}H^{-m}(h/H)\varepsilon & \text{if there is no } U^3 \text{ derivative and } i = 3, \\ (\theta/\theta_0)^{m}H^{-m}\varepsilon(H/h) & \text{otherwise,} \end{cases}$$

in the region M, and $m = 0,1,\cdots,N$.

From (3.1 - 3.3) it follows that $\frac{H}{H^*}\sqrt{\varepsilon} < \theta_0$ and hence that $\theta < \theta_0$. Setting $\Omega = \frac{h\theta}{H\theta_0}$, we have $\Omega \le \frac{\theta}{\theta_0} < 1$.

We define the wave length λ, and ratio Ω as

$$(4)\qquad \lambda = \frac{\theta_0 2H}{\theta} = \theta_0 \min(D, \sqrt{2H^*R}, \frac{2H^*}{\sqrt{\varepsilon}}), \quad \Omega = \frac{2h}{\lambda}.$$

Assumptions (3.1) imply that

$$(5)\qquad \operatorname{spectrum}(\varepsilon^{i}_{k}) < \varepsilon,\quad S^{\alpha\pm}{}_{,\alpha_1\cdots\alpha_m} = \mathcal{O}(\frac{\varepsilon\Omega}{\lambda^m}),$$

$$S^{3+}-S^{3-}{}_{,\alpha_1\cdots\alpha_m} = \mathcal{O}(\frac{\varepsilon\Omega^2}{\lambda^m})$$

$$S^{3\pm}{}_{,\alpha_1\cdots\alpha_m} = \mathcal{O}(\frac{\varepsilon}{\lambda^m}),$$

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[5] We follow Sensenig [6].

$$B^i{}_{,k_1\cdots k_{m-1}} = \begin{cases} (\frac{\varepsilon}{\lambda^m}) & \text{if } i = 3 \text{ and the number of } U^3 \text{ differentiations is odd, or if } i \neq 3 \text{ and the number of } U^3 \text{ differentiations is even} \\ (\frac{\varepsilon\Omega}{\lambda^m}) & \text{if there is no differentiation with respect to } U^3 \text{ and } i = 3, \\ (\frac{\varepsilon}{\lambda^{m-1}h}) & \text{otherwise,} \end{cases}$$

for $m = 0,1,\cdots,N$.

Restricting θ_0 by $\theta_0 < \frac{1}{2}$ we have

$$(6) \qquad 2H < \lambda < \sqrt{\frac{H^*R}{2}}, \quad 8H\frac{H}{H^*} < R, \quad \lambda \leq D/2$$

$$\frac{1}{R} \leq \frac{H^*}{2\lambda^2} \leq \frac{H^*}{H}\theta\frac{1}{\lambda} \leq \frac{\Omega}{\lambda}, \quad \text{and} \quad \varepsilon < \frac{(2H^*)^2}{\lambda^2} \leq \frac{(2H)^2}{\lambda^2}.$$

Since f and its first derivatives vanish at P_0, we have in a region $M: U^\alpha U^\alpha < \lambda^2$ by (2.3a) $|f_{,\alpha}| \leq \sqrt{2}\lambda/R$. By (2.3c) and the inequality $||h_{,\alpha}|-|f_{,\alpha}|| \leq |h_{,\alpha}-f_{,\alpha}| < \sqrt{h/R}$ we have $|h_{,\alpha}| \leq \sqrt{h/R} + \sqrt{2}\lambda/R$ at some point in the region $U^\alpha U^\alpha < 2H^*R \leq \lambda^2$. Since by (2.3b) $|h_{,\alpha\beta}| < R^{-1}$, we have

$$(7) \qquad |h_{,\alpha}| = \mathcal{O}(\sqrt{h/R} + \lambda/R) = \mathcal{O}(\Omega\theta).$$

By (2.3) we have employing (3.6), (3.7)

$$|E_{\alpha\beta} - \delta^\alpha_\beta| \leq 2\lambda^2/R^2,$$

$$|L_{\alpha\beta}| \leq 1/R.$$

In addition we have

$$E_{\alpha\beta,\gamma} = \mathcal{O}(\lambda/R^2), \; E_{\alpha\beta,\gamma\delta} = \mathcal{O}(1/R\sqrt{hR}), \; E_{\alpha\beta,\gamma\delta\mu} = \mathcal{O}(1/RhR), \ldots,$$

$$L_{\alpha\beta,\gamma} = \mathcal{O}(1/R\sqrt{hR}), \; L_{\alpha\beta,\gamma\delta} = \mathcal{O}(1/RhR), \ldots.$$

We have from (2.1) and (2.3)

$$(8) \qquad \eta = \mathcal{O}(h/R), \quad \eta' = \mathcal{O}(1/R), \quad \eta'' = \mathcal{O}(1/R\sqrt{hR}), \ldots,$$

$$\dot{\eta} = \mathcal{O}(h/R\sqrt{hR}), \quad \ddot{\eta} = \mathcal{O}(h/RhR), \ldots .$$

If $\theta_0 < \frac{1}{2}$ we have by (2.1)

$$G = (\det E_{\alpha\beta})(1-U^3 L^{\alpha}_{\alpha}+(U^3)^2 \det L^{\beta}_{\alpha})^2 > 0,$$

hence G^{ik} can be estimated by

$$G^{ik} - \delta^i_k = \mathcal{O}(|\eta|) = \mathcal{O}(h/R).$$

For the Christoffel vectors $\Gamma = (\Gamma^i_{jk})$, we have

$$(9) \qquad \Gamma = \mathcal{O}(1/R), \quad \Gamma' = \mathcal{O}(1/R\sqrt{hR}), \quad \Gamma'' = \mathcal{O}(1/RhR), \ldots .$$

Our assumption of the smallness of the principal strains furnishes the estimate $\varepsilon^i_k = \mathcal{O}(\theta_0^2)$ for the individual elements ε^i_k, provided θ_0 is sufficiently small. We have that $\varepsilon^i_k = G^{im}\varepsilon_{mk}$, G^{im} positive definite, $\varepsilon_{mk} = \varepsilon_{km}$. It is sufficient to show that ε^i_k is similar to a symmetric matrix. The similarity transformation is given by the symmetric positive matrix whose square equals G^{im}. For θ_0 sufficiently small, G^{im} is close to δ^i_j, and such a transformation can be found.

The strain energy density function $W(s_1,s_2,s_3)$ which we consider to be as regular as needed for sufficiently small $|s_i|$--the absolute value of the sum of the i-powers of the principal strains--determines the stress strain relations. It follows that

$$t^m_i = \mathcal{O}\left(\sqrt{\varepsilon^i_k \varepsilon^k_i}\right).$$

Hence, given the function W, bounds t_0, η_0 such that $|t| < t_0$ and $|\eta| < \eta_0$ can be obtained and (1.9),...,(1.17) are valid provided θ_0 is sufficiently small.

We may also conclude that the region

$$M = \{(U^1,U^2,U^3)\,|\,U^\alpha U^\alpha < \lambda^2,\ |U^3| < h\}$$

lies entirely within the shell. The distance between any point on Σ_0 and P_0 is given by

$$\sqrt{U^\alpha U^\alpha+f^2} < \sqrt{\lambda^2+4\lambda^4R^{-2}} = \lambda\sqrt{1+4\lambda^2R^{-2}}.$$

If we restrict $\theta_0 < \frac{1}{4}$ we have

$$\sqrt{U^\alpha U^\alpha+f^2} < \frac{D}{\sqrt{2}}$$

Hence for any point in M, the distance to P_0 is at most $\frac{D}{\sqrt{2}} + h < \left(\frac{\sqrt{2}}{2} + \frac{1}{4}\right)D < D$, provided $\theta_0 < \frac{1}{4}$.

In what follows, we shall use ε to denote ε_0, since no confusion will result.

Chapter 4 - Estimates for the L_2 Norms of the Second Order Derivatives of the Stresses

4.1 Coordinate Transformation to a Shell of "Constant" Thickness

We consider the transformation of the Lagrange coordinates $U = (U^1, U^2, U^3)$ to the new coordinates $\underline{U} = (\underline{U}^1, \underline{U}^2, \underline{U}^3)$ given by

(1)
$$\underline{U}^\alpha = U^\alpha,$$
$$\underline{U}^3 = \frac{H}{h(U^1,U^2)} U^3.$$

Under this transformation, the center plane of the shell Σ_0 remains fixed and the faces $U^3 = \pm h(U^1,U^2)$ are transformed to faces $\Sigma_{\pm H}$ a constant distance H from Σ_0.

For the moment, we choose to regard the t^{ik} as tensors, and calculate their new components $\hat{t}^{ik}$ in the $\underline{U}^i$ coordinate system. From the transformation formulas

$$\hat{t}^{ik} = \frac{\partial \underline{U}^i}{\partial U^\ell} \frac{\partial \underline{U}^k}{\partial U^m} t^{\ell m} \quad \text{and} \quad t^{\ell m} = \frac{\partial U}{\partial \underline{U}^i} \frac{\partial U^m}{\partial \underline{U}^k} \hat{t}^{ik}, \quad \text{we have}$$

(2)
$$t^{\alpha\beta} = \hat{t}^{\alpha\beta},$$
$$t^{3\alpha} = \frac{h(\underline{U}^1,\underline{U}^2)}{H} \hat{t}^{3\alpha} + h_{,\beta} \frac{\underline{U}^3}{H} \hat{t}^{\alpha\beta},$$
$$t^{33} = \left(\frac{h}{H}\right)^2 \hat{t}^{33} + \frac{2h}{H} h_{,\alpha} \frac{\underline{U}^3}{H} \hat{t}^{3\alpha} + h_{,\alpha} h_{,\beta} \left(\frac{\underline{U}^3}{H}\right)^2 \hat{t}^{\alpha\beta}$$

We define

$$\underline{t}^{33} = \left(\frac{h}{H}\right)^2 \hat{t}^{33}, \quad \underline{t}^{3\alpha} = \frac{h}{H} \hat{t}^{3\alpha}, \quad \underline{t}^{\alpha\beta} = \hat{t}^{\alpha\beta}, \quad \underline{u} = \frac{\underline{U}^3}{H}$$

and note that $\frac{\partial h}{\partial U^{\alpha}} = \frac{\partial h}{\partial \underline{U}^{\alpha}}$. Consequently, (4.2) takes the form

$$(3) \qquad \begin{aligned} t^{\alpha\beta} &= \underline{t}^{\alpha\beta}, \\ t^{3\alpha} &= \underline{t}^{3\alpha} + (h_{,\underline{\beta}}\underline{u})\underline{t}^{\beta\alpha}, \\ t^{33} &= \underline{t}^{33} + (2h_{,\underline{\beta}}\underline{u})\underline{t}^{\beta 3} + (h_{,\underline{\alpha}}\underline{u}^{2})\underline{t}^{\beta\alpha}, \end{aligned}$$

where $\cdot_{,\underline{\alpha}} = \frac{\partial \cdot}{\partial \underline{U}^{\alpha}}$.

Using the chain rule for differentiation we have

$$(4) \qquad \frac{\partial}{\partial U^{\alpha}} = \frac{\partial}{\partial \underline{U}^{\alpha}} - \left(\frac{H}{h}\right) h_{,\underline{\alpha}}\underline{u}\frac{\partial}{\partial \underline{U}^{3}}, \quad \frac{\partial}{\partial U^{3}} = \frac{H}{h}\frac{\partial}{\partial \underline{U}^{3}}.$$

Denoting differentiation with respect to $\underline{U}^{\alpha}$ by the symbol "$,\underline{\alpha}$" and the operator $\frac{H}{h}\frac{\partial}{\partial \underline{U}^{3}}$ by "$,\underline{3}$" we write (4.4) as

$$(5) \qquad \cdot_{,k} = \cdot_{,\underline{k}} - \cdot_{,\underline{3}}\delta^{\beta}_{k}(h_{,\underline{\beta}}\underline{u}); \quad \cdot_{,3} = \cdot_{,\underline{3}}.$$

For the Laplace operator

$$\begin{aligned} \cdot_{,ii} &= \cdot_{,\alpha\alpha} + \cdot_{,33} = \cdot_{,\alpha\alpha} + \cdot_{,\underline{33}}, \quad \text{where} \\ \cdot_{,\alpha\alpha} &= \cdot_{,\underline{\alpha\alpha}} - \cdot_{,\underline{3\alpha}}(2h_{,\underline{\alpha}}\underline{u}) + \cdot_{,\underline{33}}(h_{,\underline{\alpha}}h_{,\underline{\alpha}}\underline{u}^{2}) - \cdot_{,\underline{3}}(h_{,\underline{\alpha\alpha}}\underline{u}). \end{aligned}$$

We remark that operators "$,\underline{3}$" and "$,\underline{\alpha}$" are not commutative.

Transforming equations (1.16), where t^{ik} transforms as a contravariant tensor and the derivatives transform according to the chain rule, we have in our notation:

$$(6) \qquad t^{\alpha k}{}_{,k} = \underline{t}^{\alpha k}{}_{,\underline{k}} = -B^{\alpha} + P^{\alpha},$$

$$(7) \qquad \begin{aligned} t^{3k}{}_{,k} &= \underline{t}^{3k}{}_{,\underline{k}} - 2h_{,\underline{\alpha}}\left(\frac{1}{h}\right)\underline{t}^{3\alpha} + V + (-B^{3}+P^{3}) = -B^{3} + P^{3}, \\ V &= h_{,\underline{\alpha}}h_{,\underline{\beta}}\frac{\underline{u}}{h}\underline{t}^{\alpha\beta} + h_{,\underline{\alpha\beta}}\underline{u}\underline{t}^{\alpha\beta} + h_{,\underline{\alpha}}\underline{u}(-B^{\alpha}+P^{\alpha}) - (-B^{3}+P^{3}). \end{aligned}$$

Using (1.17), the "Beltrami-Michell" equations, and the fact that

$$t^{33} - (h_{,\rho}\underline{u})t^{\rho 3} = \underline{t}^{33} + (h_{,\underline{\rho}}\underline{u})\underline{t}^{\rho 3}$$

we have

$$\begin{aligned}(8)\quad & \underline{t}^{3k}{}_{,\underline{ii}} + \frac{1}{1+\nu}(t^{ii}{}_{,3\underline{k}} - (h_{,\underline{\rho}}\underline{u})t^{ii}{}_{,\rho\underline{k}}) = J_k = \\ & [\underline{t}^{3k}{}_{,3\underline{\delta}}(h_{,\underline{\delta}}\underline{u}) - \underline{t}^{3k}{}_{,\underline{33}}(h_{,\underline{\delta}}h_{,\underline{\delta}}\underline{u}^2) - J_0^k] - J_1^k - \\ & \delta_3^k[(h_{,\underline{\rho}}\underline{u})\underline{t}^{3\rho}{}_{,ii} + J_2^3] - \delta_k^\alpha(h_{,\underline{\alpha}}\underline{u})[\underline{t}^{33}{}_{,ii} + (h_{,\underline{\rho}}\underline{ut}^{3\rho})_{,ii} + J_2^3] + \\ & \{Q_{3k} - B_{3k} + (h_{,\underline{\rho}}\underline{u})(Q_{\rho k} - B_{\rho k}) + \\ & \delta_k^\alpha(h_{,\underline{\alpha}}\underline{u})[Q_{33} - B_{33} - (h_{,\underline{\rho}}\underline{u})(Q_{\rho 3} - B_{\rho 3})]\},\end{aligned}$$

where $B_{hk} = \frac{1}{1-\nu}(B^j{}_{,j})\delta_{hk} + B^h{}_{,k} + B^k{}_{,h}$, and J_i^k contain at most first order derivatives of the stresses and are given by

$$J_0^k = (h_{,\underline{\delta\delta}}\underline{u})\underline{t}^{3k}{}_{,\underline{3}},\quad J_1^k = (h_{,\underline{\delta}}\underline{u})_{,i}\underline{t}^{\delta k}{}_{,i} + (h_{,\underline{\delta}}\underline{u})_{,ii}\underline{t}^{\delta k},$$

$$J_2^3 = (h_{,\underline{\delta}}\underline{u})_{,i}t^{\delta 3}{}_{,i} + (h_{,\underline{\delta}}\underline{u})_{,\underline{ii}}t^{\delta 3}.$$

Derivatives of the stresses fall into two classes which are dependent on the number of 3's appearing among the superindices and subindices. We denote the N-order derivative of the stress vector $t^{ik}{}_{,n_1 \ldots n_N}$ by $t^{(\mu)(s)}{}_{,n}$ where u = number of 3's among superindices, s = number of 3's among the differentiations, and define

$$t^{(0)(s)}_{,N} = \left(\sum t^{\alpha\beta}_{,3\ldots3\delta_{s+1}\ldots\delta_N} t^{\alpha\beta}_{,3\ldots3\delta_{s+1}\ldots\delta_N}\right)^{1/2},$$

$$(9) \quad t^{(1)(s)}_{,N} = \left(\sum t^{3\beta}_{,3\ldots3\delta_{s+1}\ldots\delta_N} t^{3\beta}_{,3\ldots3\delta_{s+1}\ldots\delta_N}\right)^{1/2},$$

$$t^{(2)(s)}_{,N} = \left(\sum t^{33}_{,3\ldots3\delta_{s+1}\ldots\delta_N} t^{33}_{,3\ldots3\delta_{s+1}\ldots\delta_N}\right)^{1/2},$$

where $\mu = 0,1$ or 2 and $0 \leq s \leq N$, and we sum over all free indices. For any vector $w = (w^{33\ldots3\alpha\beta\ldots\delta})$ we can define, (where μ = number of 3's appearing in superscript)

$$w^{(\mu)(s)}_{,N} = \left(\sum w^{33\ldots3\alpha\beta\ldots\delta}_{,3\ldots3\delta_{s+1}\ldots\delta_N} w^{33\ldots3\alpha\beta\ldots\delta}_{,3\ldots3\delta_{s+1}\ldots\delta_N}\right)^{1/2}$$

and

$$w^{(\mu)}_{,N} = \sum_{s=0}^{N} w^{(\mu)(s)}_{,N}, \quad w^{e}_{,N} = \sum_{\substack{\mu-s\geq 0 \\ \text{or } \mu-s \text{ even}}} w^{(\mu)(s)}_{,N}.$$

Using the rules of transformation given by (4.3) we have for the second order contravariant tensor t^{ik}:

$$(10) \quad t^{(\mu)} = \mathcal{O}\left(\sum_{j=0}^{\mu} \Lambda^{j} \underline{t}^{(\mu-j)}\right) = \mathcal{O}\left(\underline{t}^{(\mu)} + \sum_{j=1}^{\mu} \Lambda^{j} \underline{t}^{(\mu-j)}\right).$$

We note that by (3.6a)

$$(11) \quad \Lambda = (h_{,\underline{\delta}} h_{,\underline{\delta}} u^2)^{1/2} = \mathcal{O}(\Omega\theta), \quad h_{,\underline{\delta}}/h = \mathcal{O}(\lambda^{-1}),$$

$$h_{,\underline{\alpha\beta}} u = \mathcal{O}(\Omega\theta/\lambda).$$

We define for a multi index $n_1\ldots n_N = n$ a function $p_i(N)$, $i = 1,\ldots,N!$ the set of all possible permutations of the multi index n. Obviously for any function $w \in C^N$

$w^{(s)}_{,n} = \mathcal{O}\left(w^{(s)}_{,p_i(N)}\right) = \mathcal{O}\left(w^{(s)}_{,N}\right)$ for $s = 0,\ldots,N$, and $i = 1,\ldots,N!$, since differentiation in the U^i coordinate system is commutative.

Differentiation with respect to $\underline{U}^\alpha$ and $\underline{U}^3$ is not commutative. We have, using the chain rule for $w \in C^2$:

$$w^{(2)}_{,2} = \mathcal{O}\left(w^{(2)}_{,\underline{2}}\right),$$

$$w^{(0)}_{,2} = \mathcal{O}\left(w^{(0)}_{,\underline{2}} + \left(\Lambda w^{(1)}_{,\underline{1}}\right)^{(0)}_{,1} + \Lambda\left(w^{(0)}_{,\underline{1}}\right)^{(1)}_{,\underline{1}} + \Lambda\left(\Lambda w^{(1)}_{,\underline{1}}\right)^{(1)}_{,\underline{1}}\right).$$

Noting, however, that

$$\left(w^{(0)}_{,\underline{1}}\right)^{(1)}_{,\underline{1}} = \mathcal{O}\left(\left(w^{(1)}_{,\underline{1}}\right)^{(0)}_{,\underline{1}} + \Lambda^{(1)}_{,\underline{1}} w^{(1)}_{,\underline{1}}\right),$$

we have

$$w^{(0)}_{,2} = \mathcal{O}\left(w^{(0)}_{,\underline{2}} + \left(\Lambda w^{(1)}_{,\underline{1}}\right)^{(0)}_{,\underline{1}} + \Lambda w^{(1)}_{,\underline{2}} + \Lambda\left(\Lambda w^{(1)}_{,\underline{1}}\right)^{(1)}_{,\underline{1}}\right).$$

For $w^{(1)}_{,2}$ we have by a similar calculation

$$w^{(1)}_{,2} = \mathcal{O}\left(w^{(1)}_{,\underline{2}} + \Lambda w^{(2)}_{,\underline{2}}\right) = \mathcal{O}\left(\left(w^{(0)}_{,\underline{1}}\right)^{(1)}_{,\underline{1}} + \left(\Lambda w^{(1)}_{,\underline{1}}\right)^{(1)}_{,\underline{1}}\right).$$

These results can be summarized by

$$w^{(s)}_{,2} = \mathcal{O}\left(w^{(s)}_{,\underline{2}} + \sum_{\ell=1}^{2-s}(\Lambda^{\ell}w)^{(\ell+s)}_{,\underline{2}}\right) = \mathcal{O}\left(w^{(s)}_{,p_i(2)} + \sum_{\ell=1}^{2-s}(\Lambda^{\ell}w)^{(\ell+s)}_{,\underline{2}}\right),$$

for $i = 1,2$ and $s = 0,1,2$.

Combining the results of (4.5) and (4.10) we have

$$(12)\quad t^{(\mu)(s)}_{,N} = \mathcal{O}\left(\underline{t}^{(\mu)(s)}_{,\underline{N}} + \sum_{\ell=0}^{N-s}\sum_{j=\delta^0_\ell}^{\mu}\left(\Lambda^{\ell+j}\underline{t}^{(\mu-j)}\right)^{(\ell+s)}_{,\underline{N}}\right)$$

$$= \mathcal{O}\left(\underline{t}^{(\mu)(s)}_{,p_{\underline{i}}} + \sum_{\ell=0}^{N-s}\sum_{j=\delta^0_\ell}^{\mu}\left(\Lambda^{\ell+j}\underline{t}^{(\mu-j)}\right)^{(\ell+s)}_{,\underline{N}}\right),$$

where $i = 1,2,\ldots,N!$, $s = 0,1,\ldots,N$, $\mu = 0,1,2$.

We define

$$(13)\qquad Z^{(\mu,s)}_{,\underline{N}} = \sum_{\ell=0}^{N-s} \sum_{j=\delta^0_\ell}^{\underline{\mu}} (\Lambda^{\ell+j} \underline{t}^{(\mu-j)})^{(\ell+s)}_{,\underline{N}}$$

$$Z^{e}_{,\underline{N}} = \sum_{\substack{\mu-s\geq 0 \\ \text{or even}}} Z^{(\mu,s)}_{,\underline{N}}$$

$$B = (B^i B^i)^{1/2}, \quad B_{rh} = \mathcal{O}(B_{,1})$$

$$Q = (Q^{hk} Q^{hk})^{1/2}, \quad P = (P^i P^i)^{1/2}.$$

4.2 Estimates for the L_2 Norms of the Second Derivatives

Our first task will be to derive estimates for the L_2 norms of the first and second derivatives of the stresses in terms of the state of stress--t, assumed bounded, and the quantities P, Q, and Z. We will not at this time delve into the structure of these quantities which contain terms due to the non-linearity of the constituent equations, the geometry of the shell, and the variation in thickness.

Since we have assumed no conditions on the stresses and their derivatives on the edges of the region $\underline{M}$ given by $\underline{U}^\alpha \underline{U}^\alpha = \lambda^2$, we introduce a set of auxiliary functions ϕ_i, $i = 0,1,2,3,\ldots$ defined by

$$\phi_0, \phi_1, \phi_2 = (1-\lambda^{-2}\underline{U}^\alpha \underline{U}^\alpha)^2 = (1-\lambda^{-2}U^\alpha U^\alpha)^2,$$
$$\text{for } \underline{U}^\alpha \underline{U}^\alpha = U^\alpha U^\alpha < \lambda^2$$

$$\phi_k = (1-4^{k-2}\lambda^{-2}\underline{U}^\alpha \underline{U}^\alpha)^2 = (1-4^{k-2}\lambda^{-2}U^\alpha U^\alpha)^2,$$
$$\text{for } \underline{U}^\alpha \underline{U}^\alpha = U^\alpha U^\alpha < \lambda^2 4^{2-k}, \quad \text{where } k = 3,4,\ldots.$$

$\phi_i = 0$ otherwise for $i = 0,1,2,3\ldots.$[6]

It will be understood that ϕ will denote any particular ϕ_i when no confusion can result. It is easily seen that

$$|\phi| \leq 1, \quad |\phi^{(0)}_{,1}| = \sum_{\alpha} |\phi_{,\alpha}| = \sum_{\underline{\alpha}} |\phi_{,\underline{\alpha}}| = \mathcal{O}(\sqrt{\Phi}\lambda^{-1}) = \mathcal{O}(\lambda^{-1}),$$

$$|\phi^{(0)}_{,2}| = \sum_{\alpha\beta} |\phi_{,\alpha\beta}| = \sum_{\underline{\alpha\beta}} |\phi_{,\underline{\alpha\beta}}| = |\phi^{(0)}_{,\underline{2}}| = \mathcal{O}(\lambda^{-2}),$$

$$|\phi_{k+1,1}{}^{(0)}| = |\phi_{k+1,\underline{1}}{}^{(0)}| = \mathcal{O}(|\phi_k|\lambda^{-1}),$$

$$|\phi_{k+2,2}{}^{(0)}| = |\phi_{k+2,\underline{2}}{}^{(0)}| = \mathcal{O}(|\phi_k|\lambda^{-2}),$$

$$|\phi_{i,N}{}^{(s)}| = |\phi_{i,\underline{N}}{}^{(s)}| = 0 \text{ if } s > 0, \quad i = 0,1,2,\ldots,N = 1,2,\ldots\ldots$$

For any vector w, defined in the region $\underline{M}$ given by $\underline{U}^\alpha\underline{U}^\alpha < \lambda^2$, $|\underline{U}^3| < H$ we define the L_2 norm

$$\|w\| = \sqrt{\iiint_{\underline{M}} |w|^2 d\underline{U}^1 d\underline{U}^2 d\underline{U}^3}.$$

LEMMA.

$$(14) \quad \|\phi\underline{t}^{(0)}{}^{(0)}_{,\underline{2}}\| = \mathcal{O}(\|\phi\underline{t}^{\alpha\alpha}{}_{,\underline{\xi\xi}}\| + \sum_{\xi,\gamma} \|\phi\underline{t}^{\xi\eta}{}_{,\underline{\eta\gamma}}\| + \lambda^{-2}\|\underline{t}\|).$$

PROOF:

From the appendix (A2) we have

$$(15) \qquad \|\phi\underline{t}^{\beta\delta}{}_{,\underline{\gamma\alpha}}\| = \mathcal{O}(\|\phi\underline{t}^{\beta\delta}{}_{,\underline{\alpha\alpha}}\| + \lambda^{-2}\|\underline{t}\|).$$

From the identities:

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[6] See Sensenig [6].

$$\underline{t}^{11}{}_{,\underline{\alpha\alpha}} = \underline{t}^{1\beta}{}_{,\underline{\beta 2}} - \underline{t}^{2\beta}{}_{,\underline{\beta 2}} + \underline{t}^{\beta\beta}{}_{,\underline{22}},$$

$$\underline{t}^{12}{}_{,\underline{\alpha\alpha}} = \underline{t}^{2\beta}{}_{,\underline{\beta 2}} - \underline{t}^{\beta\beta}{}_{,\underline{21}} + \underline{t}^{1\beta}{}_{,\underline{\beta 2}},$$

$$\underline{t}^{22}{}_{,\underline{\alpha\alpha}} = \underline{t}^{2\beta}{}_{,\underline{\beta 2}} - \underline{t}^{1\beta}{}_{,\underline{\beta 1}} + \underline{t}^{\beta\beta}{}_{,\underline{11}},$$

we have

$$(16) \qquad \|\phi\underline{t}^{\gamma\delta}{}_{,\underline{\alpha\alpha}}\| = \mathcal{O}\Big(\sum_{\xi,\eta}[\|\phi\underline{t}^{\xi\alpha}{}_{,\underline{\alpha\eta}}\| + \|\phi\underline{t}^{\alpha\alpha}{}_{,\underline{\xi\eta}}\|]\Big).$$

Setting $\beta = \delta$ and summing (4.15), substituting in (4.16) gives (4.15).

We introduce the quantity $A_{,\underline{N}}$ defined by

$$(17) \quad A_{,\underline{N}} = (\underline{t}^{3k}{}_{,\underline{i\alpha_1\cdots\alpha_{N-1}}}\underline{t}^{3k}{}_{,\underline{i\alpha_1\cdots\alpha_{N-1}}})^{1/2} = \sum_{\mu=1}^{2}\sum_{s=0}^{1}\underline{t}^{(\mu)(s)}{}_{,\underline{N}}.$$

Denoting

$$\iiint_{\underline{M}}[\cdot]d\underline{U}^1 d\underline{U}^2 d\underline{U}^3 \quad \text{by} \quad \int_{\underline{M}}[\cdot]d\underline{U}, \quad \text{we have}$$

$$\|\phi A_{,\underline{2}}\|^2 = \int_{\underline{M}}\phi^2\underline{t}^{3k}{}_{,\underline{i\alpha}}\underline{t}^{3k}{}_{,\underline{i\alpha}}d\underline{U},$$

and clearly we have

$$(18) \quad \left[\|\phi\underline{t}^{(1)(0)}{}_{,\underline{2}}\|,\ \|\phi\underline{t}^{(1)(1)}{}_{,\underline{2}}\|,\ \|\phi\underline{t}^{(2)(0)}{}_{,\underline{2}}\|,\ \|\phi\underline{t}^{(1)(0)}{}_{,\underline{2}}\|\right] = \mathcal{O}(\|\phi A_{,\underline{2}}\|) = \mathcal{O}(\|\phi\underline{t}^{e}{}_{,\underline{2}}\|).$$

From equations (4.6) and (4.7)

$$t^{33}{}_{,33} = -t^{3\alpha}{}_{,3\alpha}+P^3{}_{,3}-B^3{}_{,3}, \quad t^{\alpha\beta}{}_{,\beta\gamma} = -t^{3\alpha}{}_{,3\gamma}+P^\alpha{}_{,\gamma}-B^\alpha{}_{,\gamma},$$

$$(19) \quad t^{ii}{}_{,33} = -(1+\nu)(t^{33}{}_{,\alpha\alpha}+t^{33}{}_{,33}) + (1+\nu)(Q_{33}-B_{33}),$$

$$t^{\beta\beta}{}_{,\alpha\alpha} = \left(\frac{1+\nu}{2+\nu}\right)(Q_{rr}-B_{rr}) - t^{33}{}_{,\alpha\alpha} - t^{ii}{}_{,33},$$

so that transforming the stresses to $\underline{U}$ coordinates and using (4.13)

$$(20)\qquad \|\phi \underline{t}^{33}{}_{,33}\| = \mathcal{O}(\|\phi A_{,\underline{2}}\| + \|\phi(P^3{}_{,3}+B^3{}_{,3})\| + \|\phi Z^e{}_{,\underline{2}}\|),$$

$$(21)\qquad \|\phi \underline{t}^{ii}{}_{,\underline{33}}\| = \mathcal{O}(\|\phi A_{,\underline{2}}\| + \|\phi t^{33}{}_{,\underline{33}}\| + \|\phi(Q+B_{33})\| + \|\phi Z^e{}_{,\underline{2}}\|),$$

$$(22)\qquad \|\phi \underline{t}^{\beta\beta}{}_{,\underline{\alpha\alpha}}\| = \mathcal{O}(\|\phi(Q+B_{rr}+P^3{}_{,3})\| + \|\phi A_{,\underline{2}}\| + \|\phi Z^e{}_{,\underline{2}}\|).$$

Setting $h = \alpha$, $k = \beta$ in the compatibility condition (1.17) and solving for $t^{\alpha\beta}{}_{,33}$ gives

$$(23)\qquad \|\phi \underline{t}^{\alpha\beta}{}_{,\underline{33}}\| = \mathcal{O}(\|\phi \underline{t}^{\alpha\beta}{}_{,\gamma\gamma}\|+\|\phi \underline{t}^{ii}{}_{,\underline{\alpha\beta}}\|+\|\phi(Q_{\alpha\beta}+B_{\alpha\beta})\|+\|\phi Z^e{}_{,\underline{2}}\|).$$

In summary, we have for $\displaystyle\sum_{\substack{u-s\geq 0\\ \text{or even}}} \underline{t}^{(u)(s)}{}_{,\underline{2}} = \underline{t}^e{}_{,\underline{2}}$

$$(24)\qquad \|\phi \underline{t}^e{}_{,\underline{2}}\| = \mathcal{O}(\|\phi(A_{,\underline{2}} + Z^e{}_{,\underline{2}} + B_{\alpha\beta} + B_{rr})\|.$$

By (A8) of the appendix

$$\lambda^{-1}\|\phi h \underline{t}^{(0)(1)}{}_{,\underline{2}}\| = \mathcal{O}(\lambda^{-2}\|\underline{t}\| + H\lambda^{-1}(\|\phi \underline{t}^e{}_{,\underline{2}}\|)) + \sigma(\|\phi \underline{t}^e{}_{,\underline{2}}\|).$$

The equilibrium equation (4.7) gives

$$\underline{t}^{\alpha 3}{}_{,\underline{33}} = \mathcal{O}(\underline{t}^{(0)(1)}{}_{,\underline{2}} + B_{,1} + Q + Z_{,\underline{2}}).$$

Consequently we can estimate all second derivatives of the stresses

$$(25)\qquad \lambda^{-1}\|\phi h \underline{t}_{,\underline{2}}\| = \mathcal{O}(\lambda^{-2}\|\underline{t}\| + H\lambda^{-1}\|\phi \underline{t}^e{}_{,\underline{2}}\| + \lambda^{-1}\|\phi h Z_{,\underline{2}}\|$$
$$+ \lambda^{-1}\|\phi h(B_{,\underline{1}}+Q)\|) + \sigma(\|\phi \underline{t}^e{}_{,\underline{2}}\|).$$

Using the equilibrium equations (4.6-4.7), estimates for the derivatives of $h_{,\underline{\alpha}}$ (4.11) and (A3) and (A8) we have for $\mu\text{-}s \geq 0$ and all first order derivatives

$$(26)\qquad \begin{aligned} \|\phi\underline{t}^{(\mu)(s)}_{,\underline{1}}\| &= \mathcal{O}(\lambda^{-1}\|\underline{t}\| + \|\phi(B+P)\|) + \sigma(\lambda\|\phi\underline{t}^{(\mu)(0)}_{,\underline{2}}\|), \\ \lambda^{-1}\|\phi h t_{,\underline{1}}\| &= \mathcal{O}(\lambda^{-1}\|\underline{t}\|) + \sigma(H\|\underline{t}^{e}_{,\underline{2}}\|). \end{aligned}$$

It remains to estimate $A_{,\underline{2}}$. We make use of the following identity for T,Y,g functions of $\underline{U}^{i}$, twice continuously differentiable, g a function of $\underline{U}^{\alpha}$ alone and support of g contained within the region $\underline{M}$.

$$(27)\qquad \begin{aligned} \int_{\underline{M}} gT_{,\underline{\alpha\alpha}}Y_{,\underline{ii}}\,d\underline{U} &= \int_{\underline{M}} gT_{,\underline{i\alpha}}Y_{,\underline{i\alpha}}\,d\underline{U} + \int_{\underline{M}} (gT_{,\underline{\alpha\alpha}}Y_{,\underline{3}})_{,\underline{3}}\,d\underline{U} + \\ &\int_{\underline{M}} g_{,\underline{\alpha}}T_{,\underline{\alpha i}}Y_{,\underline{i}} - g_{,\underline{i}}T_{,\underline{\alpha\alpha}}Y_{,\underline{i}} + \\ &+gY_{,\underline{3}}\left[2T_{,\underline{3\alpha}}\frac{h_{,\underline{\alpha}}}{h} - T_{,\underline{3}}\left(\left(\frac{h_{,\underline{\alpha}}}{h}\right)_{,\underline{\alpha}} + \frac{h_{,\underline{\alpha}}h_{,\underline{\alpha}}}{h^{2}}\right)\right]d\underline{U}. \end{aligned}$$

Employing (4.27) we have for $\|\phi A_{,\underline{2}}\|^{2}$ from (4.17)

$$(28)\qquad \begin{aligned} \|\phi A_{,\underline{2}}\|^{2} &= \int_{\underline{M}} \phi^{2}\underline{t}^{3k}_{,\underline{\alpha\alpha}}t^{3k}_{,\underline{ii}}\,d\underline{U} - \int_{\underline{M}} \left(\phi^{2}\underline{t}^{3k}_{,\underline{\alpha\alpha}}t^{3k}_{,\underline{3}}\right)_{,\underline{3}}d\underline{U} \\ &-\int_{\underline{M}}\Big[2\phi\phi_{,\underline{\alpha}}\underline{t}^{3k}_{,\underline{\alpha i}}\underline{t}^{3k}_{,\underline{i}} + 2\phi\phi_{,\underline{\alpha}}\underline{t}^{3k}_{,\underline{\alpha\beta}}\underline{t}^{3k}_{,\underline{\beta}} \\ &+\phi^{2}\underline{t}^{3k}_{,\underline{3}}\left[\underline{t}^{3k}_{,3}\left(\frac{h_{,\underline{\alpha}}}{h}\right)_{,\underline{\alpha}} - \underline{t}^{3k}_{,\underline{3\alpha}}\left(\frac{h_{,\underline{\alpha}}}{h}\right) + \underline{t}^{3k}_{,\underline{3}}\left(\frac{h_{,\underline{\alpha}}}{h}\right)^{2}\right]\Big]d\underline{U} \\ &= \mathcal{O}\Big(\Big|\int_{\underline{M}}\phi^{2}\underline{t}^{3k}_{,\underline{\alpha\alpha}}\underline{t}^{3k}_{,\underline{ii}}\,d\underline{U}\Big| + S_{1} + \|\phi_{,\underline{1}}\underline{t}^{e}_{,\underline{1}}\|^{2} \\ &\quad + \lambda^{-2}\|\phi\underline{t}^{e}_{,\underline{1}}\|^{2}\Big) + \sigma(\|\phi A_{,\underline{2}}\|^{2}), \end{aligned}$$

where we have used the fact that $\left(\frac{h_{,\underline{\alpha}}h_{,\underline{\alpha}}}{h^{2}}\right)^{1/2} = \left(\frac{h_{,\underline{\alpha}}h_{,\alpha}}{h^{2}}\right)^{1/2} = \mathcal{O}(\lambda^{-1})$, and

$$(29)\qquad S_1 = \left|\int_{\underline{M}} \phi^2\left(\underline{t}^{3k}{}_{,\underline{\alpha\alpha}}t^{3k}{}_{,\underline{3}}\right)_{,\underline{3}}d\underline{U}\right|.$$

Substituting for $\underline{t}^{3k}{}_{,\underline{ii}}$ in (4.28) from the equation of compatibility (4.8) we have

$$(30)\qquad \int_{\underline{M}}\phi^2\underline{t}^{3k}{}_{,\underline{\alpha\alpha}}\underline{t}^{3k}{}_{,\underline{ii}}d\underline{U} = \int_{\underline{M}}\phi^2\underline{t}^{3k}{}_{,\underline{\alpha\alpha}}J_k d\underline{U}$$

$$- \frac{1}{1+\nu}\int_{\underline{M}}\phi^2\underline{t}^{3k}{}_{,\underline{\alpha\alpha}}\left[\underline{t}^{ii}{}_{,\underline{3k}} - h_{,\underline{\rho}}\underline{u}t^{ii}{}_{,\rho\underline{k}}\right]d\underline{U}$$

$$= \mathcal{O}\left(\left|\int_{\underline{M}}\phi^2\underline{t}^{3k}{}_{,\underline{\alpha\alpha}}J_k d\underline{U}\right| + S_2\right.$$

$$\left.+ \left|\int_{\underline{M}}\left[\phi\phi_{,\underline{\beta}}\underline{t}^{3\beta}{}_{,\underline{\alpha\alpha}} + \phi^2\underline{t}^{3k}{}_{,\underline{\alpha\alpha k}}\right]\left[t^{ii}{}_{,3} - \left(h_{,\underline{\rho}}\underline{u}\right)t^{ii}{}_{,\rho}\right]d\underline{U}\right|\right),$$

$$(31)\qquad \text{where}\quad S_2 = \left|\int_{\underline{M}}\left[\phi^2\underline{t}^{33}{}_{,\underline{\alpha\alpha}}\left(t^{ii}{}_{,\underline{3}} - \left(h_{,\underline{\rho}}\underline{u}\right)t^{ii}{}_{,\rho}\right)\right]_{,\underline{3}}d\underline{U}\right|.$$

Switching the order of differentiation from $\underline{t}^{3k}{}_{,\underline{\alpha\alpha k}}$ to $\underline{t}^{3k}{}_{,\underline{k\alpha\alpha}}$ and integrating by parts with respect to $\underline{U}^{\alpha}$ gives

$$(32)\qquad \int_{\underline{M}}\phi^2\underline{t}^{3k}{}_{,\underline{\alpha\alpha}}\underline{t}^{3k}{}_{,\underline{ii}}d\underline{U} = \mathcal{O}\left(\left|\int_{\underline{M}}\left[\phi\underline{t}^{3k}{}_{,\underline{k\alpha}}\right]\phi\left(t^{ii}{}_{,\underline{3\alpha}} - \left[h_{,\underline{\rho}}\underline{u}\right]t^{ii}{}_{,\rho\underline{\alpha}}\right)\right.\right.$$

$$+ 2\phi_{,\underline{\alpha}}\left(t^{ii}{}_{,\underline{3}} - \left[h_{,\underline{\rho}}\underline{u}\right]t^{ii}{}_{,\rho}\right)d\underline{U}\bigg| + \left|\int_{\underline{M}}\phi^2\underline{t}^{3k}{}_{,\underline{\alpha\alpha}}J_k d\underline{U}\right| + S_2$$

$$+ \left|\int_{\underline{M}}\left[\phi\phi_{,\underline{\beta}}\underline{t}^{3\beta}{}_{,\underline{\alpha\alpha}} + \phi^2\left(\underline{t}^{33}{}_{,\underline{3\alpha}}\frac{h_{,\underline{\alpha}}}{h} + \underline{t}^{33}{}_{,\underline{3}}\frac{h_{,\underline{\alpha\alpha}}}{h}\right)\right]\left[t^{ii}{}_{,\underline{3}}\right.\right.$$

$$\left.\left.\left.- \left[h_{,\underline{\rho}}\underline{u}\right]t^{ii}{}_{,\rho}\right]d\underline{U}\right|\right).$$

Using the equilibrium equation (4.7) in the form

$$(33)\qquad \underline{t}^{3k}{}_{,\underline{k\alpha}} = 2\left(\frac{h_{,\underline{\beta}}}{h}\underline{t}^{3\beta}\right)_{,\underline{\alpha}} + V_{,\underline{\alpha}}$$

$$(34)\qquad \underline{t}^{33}{}_{,\underline{3}} = -\underline{t}^{3\beta}{}_{,\underline{\beta}} + 2\left(\frac{h_{,\underline{\beta}}}{h}\underline{t}^{3\beta}\right) + V,$$

and substituting in the first integral appearing on the right hand side of the estimate (4.32) we get for that integral

$$(35)\qquad \int_{\underline{M}}\left[\phi\left(2\frac{h_{,\underline{\beta}}}{h}\underline{t}^{3\beta}\right)_{,\underline{\alpha}} + V_{,\underline{\alpha}}\right]\left[\phi\left((t^{ii}h^{-1})_{,\underline{\alpha3}}h - (h_{,\underline{\rho}}\underline{u})t^{ii}{}_{,\rho\underline{\alpha}}\right)\right.$$

$$\left. + 2\phi_{,\underline{\alpha}}\left(t^{ii}{}_{,\underline{3}} - (h_{,}\underline{\ }\underline{u})t^{ii}{}_{,\rho}\right)\right]d\underline{U}$$

$$= \left(S_3 + \left|\int_{\underline{M}}\phi\left[\left(\frac{h_{,\underline{\beta}}}{h}\underline{t}^{3\beta}\right)_{,\underline{3\alpha}} + \left(\frac{h_{,\underline{\beta}}}{h}\underline{t}^{3\beta}\right)_{,\underline{3}}\left(\frac{h_{,\underline{\alpha}}}{h}\right)\right]\left[\phi(t^{ii}h^{-1})_{,\underline{\alpha}}h\right.\right.\right.$$

$$\left.\left. + 2\phi_{,\underline{\alpha}}t^{ii}\right]d\underline{U}\right| + \left|\int_{\underline{M}}\lambda h^{-1}V_{,\underline{\alpha}}h\lambda^{-1}\left[\phi\left((t^{ii}h^{-1})_{,\underline{\alpha3}}h - (h_{,\underline{\rho}}\underline{u})t^{ii}{}_{,\rho\underline{\alpha}}\right)\right.\right.$$

$$\left.\left.\left. + 2\phi_{,\underline{\alpha}}\left(t^{ii}{}_{,\underline{3}} - (h_{,\underline{\rho}}\underline{u})t^{ii}{}_{,\rho}\right)\right]d\underline{U}\right|\right), \quad \text{where}$$

$$S_3 = \left|\int_{\underline{M}}\phi\left(\frac{h_{,\underline{\beta}}}{h}\underline{t}^{3\beta}\right)_{,\underline{\alpha}}\left(\phi(t^{ii}h^{-1})_{,\underline{\alpha}}h + \phi_{,\underline{\alpha}}t^{ii}\right)_{,\underline{3}}d\underline{U}\right|.$$

For the last integral in (4.32) we have, using the equilibrium equations in the form of (4.33-4.34), that it is bounded by an expression of the form

$$(36)\quad \Big|\int_{\underline{M}} \phi\phi_{,\underline{\beta}}\underline{t}^{3\beta}{}_{,\underline{3\alpha}}t^{ii}{}_{,\underline{\alpha}} + (\phi\phi_{,\underline{\alpha}})_{,\underline{\beta}}\underline{t}^{3\beta}$$

$$- 2\phi\phi_{,\underline{\beta}}\left(\underline{t}^{3\beta}{}_{,\underline{3\alpha}}\frac{h_{,\underline{\alpha}}}{h} + \underline{t}^{3\beta}{}_{,\underline{3}}\frac{h_{,\underline{\alpha\alpha}}}{h}\right)t^{ii}$$

$$+ \phi^{2}\lambda h^{-1}\left[V_{,\underline{\alpha}}\left(\frac{h_{,\underline{\alpha}}}{h}\right) + V_{,\underline{\alpha}}\frac{h_{,\underline{\alpha\alpha}}}{h}\right]h\lambda^{-1}t^{ii}{}_{,\underline{3}}$$

$$- \left[\phi^{2}\frac{h_{,\underline{\alpha}}}{h}t^{ii}{}_{,\underline{\alpha}} + \left(2\phi\phi_{,\underline{\alpha}}\frac{h_{,\underline{\alpha}}}{h} + \phi^{2}\left(\frac{h_{,\underline{\alpha}}}{h}\right)_{,\alpha}\right)t^{ii}\right]\underline{t}^{3\beta}{}_{,\underline{3\beta}}$$

$$+ t^{ii}\phi^{2}\left[\frac{h_{,\underline{\alpha\alpha}}}{h}\left(\underline{t}^{3\beta}{}_{,\underline{3\beta}} - \frac{h_{,\underline{\beta}}}{h}\underline{t}^{3\beta}{}_{,\underline{3}}\right) - 2\frac{h_{,\underline{\alpha}}}{h}\left[\left(\frac{h_{,\underline{\beta}}}{h}\underline{t}^{3\beta}\right)_{,\underline{3\alpha}} - \left(\frac{h_{,\underline{\beta}}}{h}\underline{t}^{3\beta}\right)_{,\underline{3}}\frac{h_{,\underline{\alpha}}}{h}\right]\right.$$

$$\left. + \frac{h_{,\underline{\alpha}}}{h}\left(\frac{h_{,\underline{\beta}}}{h}\underline{t}^{3\beta}{}_{,\underline{3\alpha}} + \frac{h_{,\underline{\alpha}}}{h}\underline{t}^{3\beta}{}_{,\underline{3\beta}} + \frac{h_{,\underline{\alpha\beta}}}{h}\underline{t}^{3\beta}{}_{,\underline{3}}\right)\right]d\underline{U}\Big| + S_{4}, \quad \text{where}$$

$$S_{4} = \Big|\int_{\underline{M}}\left(\left[\phi\phi_{,\underline{\beta}}\underline{t}^{3\beta}{}_{,\underline{\alpha\alpha}} + \phi^{2}\left(\left(2\frac{h_{,\underline{\beta}}}{h}\underline{t}^{3\beta}\right)_{,\underline{\alpha}} - \underline{t}^{3\beta}{}_{,\underline{\beta\alpha}}\right)\frac{h_{,\underline{\alpha}}}{h}\right.\right.$$

$$\left.\left. + \frac{h_{,\underline{\alpha\alpha}}}{h}\left(-\underline{t}^{3\beta}{}_{,\underline{\beta}} + 2\frac{h_{,\underline{\beta}}}{h}\underline{t}^{3\beta}\right)\right]t^{ii}\right)_{,\underline{3}}d\underline{U}\Big|.$$

Substituting the estimates (4.32), (4.35), (4.36) into the estimate (4.30), and replacing t^{ii} and its derivatives by their components in the $\underline{U}$ coordinate system, we have in terms of our notation

$$(37)\quad \int_{\underline{M}}\phi^{2}\underline{t}^{3k}{}_{,\underline{\alpha\alpha}}\underline{t}^{3k}{}_{,\underline{ii}}d\underline{U} = \mathcal{O}\left(\Big|\int_{\underline{M}}\phi^{2}\underline{t}^{3k}{}_{,\underline{\alpha\alpha}}J_{k}d\underline{U}\Big|\right.$$

$$\left. + S_{2} + S_{3} + S_{4} + \lambda^{-2}\|\phi\underline{t}^{e}{}_{,\underline{1}}\|^{2} + \lambda^{-4}\|\underline{t}\|^{2} + \|\Omega^{-1}V^{(0)}_{,\underline{1}}\|^{2}\right)$$

$$+ \sigma\left(\|\Omega\phi\underline{t}_{,\underline{2}}\|^{2} + \|\phi\underline{t}^{e}{}_{,\underline{2}}\|^{2} + \lambda^{-2}\|\Lambda\phi\underline{t}_{,\underline{1}}\|^{2}\right).$$

We now estimate the term involving J_k appearing in (4.37). From (4.8) we see that J_k contains terms due to the non-linearity of the equations, the presence of body forces, the variation of thickness in the shell. Transforming all stress tensors and their derivatives to the $\underline{U}$ coordinate system, and using the identity (4.27) to interchange $\underline{i}$ and $\underline{\alpha}$ differentiations we get

$$\begin{aligned}(38)\quad \int_{\underline{M}} \phi^2 \underline{t}^{3k}{}_{,\underline{\alpha\alpha}} J_k \, d\underline{U} = \mathcal{O}\Big(&\Lambda\|\phi A_{,\underline{2}}\|^2 + \lambda^{-2}\|\phi \underline{t}^{e}{}_{,\underline{1}}\|^2 + \|\phi_{,\underline{1}}\underline{t}_{,\underline{1}}\|^2 \\ &+\lambda^{-4}\|\underline{t}\|^2 + \Big|\int_{\underline{M}}\phi^2\underline{t}^{3\beta}{}_{,\underline{\alpha\alpha}}B^{\beta}{}_{,\underline{3}}\, d\underline{U}\Big| + S_1 + S_5 + \|\phi B^{(1)(0)}_{,\underline{1}}\|^2 \\ &+ \|\phi B^{(1)(1)}_{,\underline{1}}\|^2 + \|\phi\Lambda B_{,\underline{1}}\|^2 + \|\phi Q\|^2\Big) + \sigma\big(\|\phi\underline{t}^{e}{}_{,\underline{2}}\|^2 + \|\phi\Lambda\underline{t}_{,\underline{2}}\|^2\big),\end{aligned}$$

$$\text{where}\quad S_5 = \Big|\int_{\underline{M}}\phi^2\Big[\frac{h_{,\underline{\delta}}}{h}\underline{t}^{3k}{}_{,\underline{\alpha\alpha}}\underline{t}^{k\delta}\Big]_{,3} d\underline{U}.$$

We have for the integral involving the body force

$$\begin{aligned}\int_{\underline{M}}\phi^2\underline{t}^{3\beta}{}_{,\underline{\alpha\alpha}}B^{\beta}{}_{,\underline{3}}\, d\underline{U} = \mathcal{O}\Big(&S_6 + \Big|\int_{\underline{M}}\phi^2\Big(2\underline{t}^{3\beta}{}_{,\underline{3\alpha}}\Big(\frac{h_{,\underline{\alpha}}}{h}\Big) + 2\underline{t}^{3\beta}{}_{,\underline{3}}\frac{h_{,\underline{\alpha\alpha}}}{h}\Big)B^{\beta}\, d\underline{U}\Big| \\ &+ \Big|\int_{\underline{M}}2\phi\phi_{,\underline{\alpha}}\underline{t}^{3\beta}{}_{,\underline{3\alpha}}B^{\beta} + \phi^2\underline{t}^{3\beta}{}_{,\underline{3\alpha}}B^{\beta}{}_{,\underline{\alpha}}\, dU\Big|\Big),\end{aligned}$$

$$(39)\quad \text{where}\quad S_6 = \Big|\int_{\underline{M}}\big[\underline{t}^{3\beta}{}_{,\underline{\alpha\alpha}}B^{\beta}\big]_{,\underline{3}}\, d\underline{U}\Big|.$$

Combining the results of estimates (4.26) through (4.39) and noting that $\Lambda = \mathcal{O}(\Omega\theta_0)$, $\sup_{\underline{M}} \Lambda = \mathcal{O}(\theta)$, we have

$$\begin{aligned}\|\phi A_{,\underline{2}}\|^2 = \mathcal{O}\big(&S_1 + S_2 + S_3 + S_4 + S_5 + S_6 + \theta\|\phi A_{,\underline{2}}\|^2 \\ &+ \|\phi B^{e}{}_{,\underline{1}}\|^2 + \|\phi\Omega B_{,\underline{1}}\|^2 + \|\phi(Q+\lambda h^{-1}V^{(0)}_{,\underline{1}})\|^2 + \lambda^{-2}\|t\|^2\big) \\ &+ \sigma\big(\lambda^{-2}\|\phi h\underline{t}_{,\underline{2}}\|^2 + \|\phi\underline{t}^{e}{}_{,\underline{2}}\|^2\big).\end{aligned}$$

Hence, from the estimates (4.18), (4.24), we have

$$(40)\quad \|\phi\underline{t}^e_{,\underline{2}}\|^2 = \mathcal{O}(\|\phi[Q + \lambda h^{-1}V^{(0)}_{,\underline{1}} + V_{,\underline{1}} + Z^e_{,\underline{2}} + \lambda^{-1}V]\|^2$$
$$+ \lambda^{-4}\|\underline{t}\| + \theta\|\phi\underline{t}^e_{,\underline{2}}\|^2 + \|\phi B^e_{,\underline{1}}\|^2 + \|\phi\Omega B_{,\underline{1}}\|^2$$
$$+ S_1 + S_2 + S_3 + S_4 + S_5 + S_6) + \sigma(\lambda^{-1}\|\phi h\underline{t}_{,\underline{2}}\|^2).$$

From (4.40) and (4.25) we have estimates for the L_2 norms of all possible second derivatives of the stress vector $\underline{t}$ in terms of the L_2 norms of the boundary terms, the stresses and the quantities Q, V, B and their derivatives. Substituting into estimates (4.26) we have similar estimates for all the first derivatives of $\underline{t}$.

4.3 Estimates for Boundary Terms and Body Forces Occurring in Estimates for the L_2 Norms of the Derivatives of the Stresses

Boundary terms appearing in our estimates are of the general form

$$(41)\qquad \int_{\underline{M}} (fg)_{,\underline{3}}d\underline{U} = \iint_{\underline{U}^\alpha\underline{U}^\alpha<\lambda^2} [(fg)^+ - (fg)^-]d\underline{U}^1 d\underline{U}^2,$$

where $f^+ = f\big|_{\underline{U}^3=+H}$ and $f^- = f\big|_{\underline{U}^3=-H}$.

We use the inequality for positive constants K and D:

$$(42)\quad |(f^+g^+-f^-g^-)| = \tfrac{1}{2}[(f^++f^-)(g^+-g^-) + (f^+-f^-)(g^++g^-)]$$
$$\leq K(f^++f^-)^2 + K^{-1}(g^+-g^-)^2 + D(f^+-f^-)^2 + D^{-1}(g^++g^-)^2.$$

In our estimate the quantity $S_{,\underline{N}}$ defined by

$$(43)\quad S_{,\underline{N}} = \sum_{i=0}^{N}\Big[\Big(\underline{t}^{(2)+}-\underline{t}^{(2)-}\Big)^{(0)}_{,\underline{N-i}}\Omega^{-2}\lambda^{-i}$$
$$+\Big(\underline{t}^{(1)+(0)}_{,\underline{N-i}}+\underline{t}^{(1)-(0)}_{,\underline{N-i}}\Big)\Omega^{-1}\lambda^{-i} + \Big(\underline{t}^{(2)+(0)}_{,\underline{N-i}}+\underline{t}^{(2)-(0)}_{,\underline{N-i}}\Big)\lambda^{-i}\Big],$$

for $N = 2$, will play a central role.

From (4.29) we have using (4.41), (4.42)

$$S_1 = \Big|\int_{\underline{M}}\Big(\phi^2\underline{t}^{3k}{}_{,\underline{\alpha\alpha}}\underline{t}^{3k}{}_{,\underline{3}}\Big)_{,\underline{3}}d\underline{U}\Big|$$
$$= \mathcal{O}\Big(\Big|\iint_{\underline{U}^\alpha\underline{U}^\alpha<\lambda^2}\phi^2\Big(\underline{t}^{3\beta+}{}_{,\underline{\alpha\alpha}} + \underline{t}^{3\beta-}{}_{,\underline{\alpha\alpha}}\Big)^2\Omega^{-2}d\underline{U}^1d\underline{U}^2\Big|H$$
$$+ \Big|\iint_{\underline{U}^\alpha\underline{U}^\alpha<\lambda^2}\phi^2\Big(\underline{t}^{33+}{}_{,\underline{\alpha\alpha}} + t^{33-}{}_{,\underline{\alpha\alpha}}\Big)^2 d\underline{U}^1d\underline{U}^2\Big|H$$
$$+ \Big|\iint_{\underline{U}^\alpha\underline{U}^\alpha<\lambda^2}\phi^2\Big(\underline{t}^{33+}{}_{,\underline{\alpha\alpha}} - \underline{t}^{33-}{}_{,\underline{\alpha\alpha}}\Big)^2\Omega^{-4}d\underline{U}^1d\underline{U}^2\Big|H$$
$$+ \Big|\iint_{\underline{U}^\alpha\underline{U}^\alpha\ \lambda^2}\phi^2\Big(\underline{t}^{3\beta+}{}_{,\underline{\alpha\alpha}} - \underline{t}^{3\beta-}{}_{,\underline{\alpha\alpha}}\Big)^2\Omega^{-2}d\underline{U}^1d\underline{U}^2\Big|H\Big)$$
$$+ \sigma\Big(\sum_\beta\Big(\theta^2H^{-2}\|\phi h\underline{t}^{3\beta}{}_{,\underline{33}}\|^2 + \|\phi\underline{t}^{33}{}_{,\underline{33}}\|^2 + \lambda^{-2}\|\phi\underline{t}^{3\beta}{}_{,\underline{3}}\|^2$$
$$+ \theta^2\lambda^{-2}\|\phi\underline{t}^{33}{}_{,\underline{3}}\|^2\Big)\Big)$$
$$= \mathcal{O}\Big(\|\phi S_{,\underline{2}}\|^2\Big) + \sigma\Big(\|\Omega\phi\underline{t}_{,\underline{2}}\|^2 + \|\phi\underline{t}^e{}_{,\underline{2}}\|^2 + \lambda^{-2}\|\phi\underline{t}^e{}_{,\underline{1}}\|^2\Big).$$

In order to estimate $S_2,\ldots,S_5$ we transform all stresses and their derivatives appearing in these estimates to their components in the $\underline{U}$ coordinate system. By a technique similar to that used for estimating S_1, we have from (4.31)

(44) $$S_2 = \mathcal{O}\left(\|\phi s_{,\underline{2}}\|^2\right) + \sigma\left(\lambda^{-2}\|\phi\Omega \underline{t}_{,\underline{1}}\|^2 + \|\phi \underline{t}_{,\underline{2}}\|^2 + \|\phi\Omega \underline{t}_{,\underline{2}}\|^2\right).$$

Continuing in this fashion, it is easy to show that

$$S_1 + S_2 + S_3 + S_4 + S_5 = \mathcal{O}\left(\|\phi s_{,\underline{2}}\|^2\right) + \sigma\Big(\lambda^{-4}\|\underline{t}\|^2 + \lambda^{-2}\|\phi h\underline{t}_{,\underline{2}}\|^2 + \|\phi \underline{t}^{e}_{,\underline{2}}\|^2\Big)$$

where we have estimated all the first order derivatives appearing on the right hand side by (4.26).

In order to estimate the terms involving the body forces, we define

(45) $$B_{,N} = \sum_{i=0}^{N} \lambda^{-i}\left(B^k{}_{,j_1\ldots j_{N-i}} B^k{}_{,j_1\ldots j_{N-i}}\right)^{1/2}$$

From our assumption on the body forces given in terms of (3.5) and the estimate $\Lambda = \mathcal{O}(\Omega\theta_0)$ we conclude that

(46) $$\left(B^{(\mu)(s)}_{,N-1};\ B^{(\mu)(s)}_{,\underline{N-1}}\right) = \begin{cases} \mathcal{O}(\varepsilon\lambda^{-N}) & \text{if } \mu-s \text{ is even,} \\ \mathcal{O}(\varepsilon\lambda^{-N}(\theta_0 h/\lambda)) & \text{if } \mu-s > 0, \end{cases}$$

$$\left(hB_{,N-1};\ hB_{,\underline{N-1}}\right) = \mathcal{O}(\varepsilon\lambda^{-N+1}) \qquad \text{otherwise.}$$

Consequently,

(47) $$\Big(\|\phi B^{(0)}_{,1}\|,\ \|\phi\Omega B_{,1}\|,\ \|\phi B^{(0)}_{,\underline{1}}\|,\ \|\phi\Omega B_{,\underline{1}}\|,\ \lambda\|\phi h^{-1}B^{(1)(0)}_{,\underline{1}}\|,\ \|\phi B^{(1)(1)}_{,1}\|,\ \|\phi B^{(1)(1)}_{,\underline{1}}\|\Big) = \mathcal{O}(\varepsilon\lambda^{-2}\|\phi\|).$$

For the boundary term S_6 we have from (4.39)

$$S_6 = \mathcal{O}\left(\|\phi s_{,\underline{2}}\|^2 + (\varepsilon\lambda^{-2})^2\|\phi\|^2\right).$$

4.4 Estimates for Terms Arising from the Transformation of Lagrange Coordinate Systems

For the expression V appearing in the equilibrium equation (4.7) we have by (4.46)

$$\lambda\|\phi h^{-1}V^{(0)}_{,\underline{1}}\| = \mathcal{O}(\lambda\|\phi h^{-1}(B^{(0)}+P)^{(0)}_{,\underline{1}}\| + \lambda^{-1}\|\phi(B+P)\| + \lambda^{-2}\|\underline{t}\|)$$

$$(49) \qquad = \mathcal{O}(\|\phi P^{(0)}_{,\underline{1}}\| + \lambda^{-2}\|\phi P\| + \lambda^{-2}\|\underline{t}\| + (\varepsilon/\lambda^2)\|\phi\|),$$

$$\|\phi V_{,\underline{1}}\| = \mathcal{O}(\lambda^{-1}\|\phi P\| + \|\phi P_{,\underline{1}}\| + \lambda^{-2}\|\underline{t}\|) + \sigma(\|\phi\underline{t}^{e}_{,\underline{2}}\|).$$

Investigating the terms $Z^{(\mu,s)}_{,\underline{2}}$ (4.13) which absorb the terms induced by the change of Lagrange coordinate systems, we find that if $(\mu-s)$ is even, the general term in $Z^{e}_{,\underline{N}}$ is of the form

$$(\Lambda^{\ell+j})^{(M_1)}_{,\underline{N_1}} \cdot \underline{t}^{(\mu-j)}{}^{(M_2)}_{,\underline{N_2}}$$

where $M_1 \leq N_1$, $M_2 \leq N_2$, $N_1+N_2 = N$, $M_1+M_2 = \ell+s$. If $(\mu-j)-M_2$ is odd, we have $(\ell+j)-M_1$ is odd. For if not, adding, we have $\mu + \ell - M_1 - M_2 = \mu-s$, which is even, a contradiction. Similarly, if $(\mu-j)-M_2$ is even, then $(\ell+j)-M_1$ is even. In addition, we note that all N-order derivatives in $Z_{,\underline{N}}$ have at least one factor of Λ multiplying them, Next, we observe from (4.10) that $\Lambda^{(2)}_{,\underline{2}} = \Lambda^{(2)}_{,2} = 0$. We have for $s \leq M$, $N > 0$:

$$\Lambda^{N}_{,M} = \Lambda^{N\,(s)}_{,\underline{M}} = 0 \qquad \text{if } s > N,$$

$$(\Lambda^{N\,(s)}_{,M},\ \Lambda^{N\,(s)}_{,\underline{M}}) = \mathcal{O}((\Omega\theta_0)^{N-s}\lambda^{-M}) \qquad \text{otherwise.}$$

Hence if $N-s$ is odd, we have

$$\left(\Lambda^{N}{}^{(s)}_{,\underline{M}} ; \Lambda^{N}{}^{(s)}_{,M}\right) = \mathcal{O}(\Omega\theta_0\lambda^{-M}).$$

Consequently if $Z^{(\mu,s)}_{,\underline{N}}$ is such that $(\mu-s)$ is even or greater than zero, then every term in the sum will be majorized by some expression of the form

$$\theta_0\Omega\lambda^{-N_1}\underline{t}^{(x)}{}^{(y)}_{,\underline{N_2}} \quad \text{for} \quad x-y < 0 \text{ and odd,} \qquad \text{or}$$

$$\lambda^{-N_1}\underline{t}^{(x)}{}^{(y)}_{,\underline{N_2}} \qquad \text{for} \quad x-y \geq 0 \text{ or even, where} \quad N_1+N_2 = N.$$

In addition, all N-order derivatives of the stresses $\underline{t}$ appearing in $Z^{e}_{,\underline{N}}$ will have at least one power of $\Omega\theta_0$ multiplying them.

In particular, we have for $N = 2$

$$(50) \quad \|\phi Z^{e}_{,\underline{2}}\| = \mathcal{O}(\|\phi\Omega\theta_0\underline{t}_{,\underline{2}}\| + \lambda^{-1}\|\phi\underline{t}^{e}_{,\underline{1}}\| + \lambda^{-1}\|\phi\Omega\theta_0\underline{t}_{,\underline{1}}\| + \lambda^{-2}\|\underline{t}\|).$$

4.5 Analysis of the Non-Linear Terms Occurring in the Estimates for the Derivatives of the Stresses

Transforming all the tensors t and derivatives to the Lagrangian coordinate system $\underline{U}^{i}$ we have:

$$t = F(\Lambda)[\underline{t}],$$

$$t^{\cdot} = t^{(0)}_{,1} = F(\Lambda)[\underline{t}^{(0)}_{,\underline{1}} + \Lambda^{(0)}_{,\underline{1}}\underline{t} + \Lambda\underline{t}_{,\underline{1}}]$$

$$t^{(1)}_{,1} = F(\Lambda)[\underline{t}^{(1)}_{,1} + \Lambda^{(1)}_{,\underline{1}}\underline{t}],$$

$$t' = t_{,1} = F(\Lambda)[\underline{t}_{,\underline{1}} + \Lambda_{,\underline{1}}\underline{t}],$$

where F stands for a different expression in each case.

On the basis of our assumptions on the curvature of the undeformed midsurface we have from (3.8)

$$\eta = \mathcal{O}(\theta_0 \Omega H^*/\lambda), \eta_{,1} = \mathcal{O}(\theta_0 H^*/\lambda^2), \ldots, \eta_{,N} = \mathcal{O}(\theta_0 H^*/\lambda^{N+1}),$$

$$(51a) \quad \eta_{,\underline{1}} = \mathcal{O}(\theta_0 H^*/\ ^2), \ldots, \eta_{,\underline{N}} = \mathcal{O}(\theta_0 H^*/\lambda^{N+1}),$$

$$\eta^{(0)}_{,1} = \mathcal{O}(\theta_0 \Omega H^*/\lambda^2), \ldots, \eta^{(0)}_{,N} = \mathcal{O}(\theta_0 \Omega H^*/\lambda^{N+1}),$$

$$\eta^{(0)}_{,\underline{1}} = \mathcal{O}(\theta_0 \Omega H^*/\lambda^2), \ldots, \eta^{(0)}_{,\underline{N}} = \mathcal{O}(\theta_0 \Omega H^*/\lambda^{N+1}).$$

For convenience we also list:

$$\left(\eta^{(0)}_{,1}\eta^{(1)}_{,1};\ \left(\eta^{(1)}_{,1}\right)^2\right) = \mathcal{O}(\theta_0^2 H^{*2}/\lambda^4),$$

$$(51b) \quad \left(\Lambda^{(0)}_{,2};\ \Lambda^{(1)}_{,2}; \left(\Lambda_{,1}\right)^2\right) = \mathcal{O}(\theta_0 \lambda^{-2}),$$

$$\left(\eta^{(0)}_{,N};\ \eta^{(1)}_{,N};\ \eta^{(s)}_{,N}\right) = \mathcal{O}(\theta_0 H^*/\lambda^{N+1}), \quad 1 \le s \le N,$$

$$\left(\eta\Lambda_{,1}\right) = \mathcal{O}(\theta_0 \Omega H^*/\lambda^2), \left(\eta_{,1}\Lambda_{,1}; \left(\eta_{,1}\right)^2;\ \eta\Lambda^2_{,1}\right) = \mathcal{O}(\theta_0 H^*/\lambda^2).$$

These relations also hold if all derivatives are with respect to the corresponding $\underline{U}$ coordinates. This is easily established by use of the chain rule.

Transforming P^i and Q^{hk} to $\underline{U}^i$ coordinate system, and using the estimates (4.51) we have from (1.17)

$$P^i = F(\eta, \Lambda, \underline{t})[\underline{t}\underline{t}_{,\underline{1}} + \Lambda_{,\underline{1}}\underline{t}^2 + \eta\underline{t}_{,\underline{1}} + \eta\Lambda_{,\underline{1}}\underline{t}]$$

$$(52) \qquad P^i = F(\eta, \Lambda, \underline{t})[C_1\underline{t}_{,\underline{1}} + C_2\underline{t}], \quad \text{where} \quad C_1 = \mathcal{O}(\theta_0^2 H^* h/\lambda^2)$$

$$\text{and} \quad C_2 = \mathcal{O}(\theta_0^2 H^* h/\lambda^3).$$

$$Q = F(r,\Lambda,\underline{t})[\underline{t}\underline{t}_{,\underline{2}} + \Lambda_{,\underline{1}}\underline{t}_{,\underline{1}}\underline{t} + (\Lambda_{,\underline{1}})^2\underline{t}^2 + r_{,\underline{1}}\underline{t}_{,\underline{1}} + r_{,\underline{1}}\Lambda_{,\underline{1}}\underline{t} + (r_{,1})^2\underline{t} + (\underline{t}_{,\underline{1}})^2 + r_{,\underline{2}}\underline{t} + r\underline{t}_{,\underline{2}} + r\Lambda_{,\underline{1}}\underline{t}_{,\underline{1}} + \Lambda_{,\underline{2}}\underline{t}^2 + r\Lambda_{,\underline{2}}\underline{t}]. \tag{53}$$

Differentiating (4.52) with respect to $\underline{U}^\alpha$ we have

$$P^{(0)}_{,\underline{1}} = F(r,\Lambda,\underline{t})[(\underline{t}+r)\underline{t}_{,\underline{2}} + (\Lambda_{,\underline{1}}\underline{t} + \underline{t}_{,\underline{1}} + r_{,\underline{1}} + r\Lambda_{,\underline{1}})t^{(0)}_{,\underline{1}} + (\Lambda^{(0)}_{,\underline{1}}\underline{t} + r^{(0)}_{,\underline{1}})\underline{t}_{,\underline{1}} + (r^{(0)}_{,\underline{1}}\Lambda_{,\underline{1}} + r\Lambda_{,\underline{2}} + r_{,\underline{2}} + r_{,\underline{1}}r^{(0)}_{,\underline{1}} + \Lambda_{,\underline{2}}\underline{t})\underline{t}]. \tag{53a}$$

From the estimate (4.51) we have

$$\lambda h^{-1}P^{(0)}_{,1} = F(r,\Lambda,\underline{t})[C_1\underline{t}_{,\underline{2}} + C_2 t^{(0)}_{,\underline{1}} + \lambda h^{-1}\underline{t}_{,\underline{1}}t^{(0)}_{,\underline{1}} + C_3\underline{t}_{,\underline{1}} + C_4\underline{t}], \text{ where } C_1 = \mathcal{O}(\theta_0 H^*/\lambda),$$

$$C_2 = \mathcal{O}(1/\lambda), \quad C_3 = \mathcal{O}(\theta_0 H^*/\lambda^2) \text{ and } C_4 = \mathcal{O}(1/\lambda^2).$$

For a general derivative of P we have

$$\begin{aligned} P_{,\underline{1}} &= F(r,\Lambda,\underline{t})[\underline{t}\underline{t}_{,\underline{2}} + \Lambda_{,\underline{1}}\underline{t}_{,\underline{1}}\underline{t} + (\Lambda_{,\underline{1}})^2\underline{t}^2 + r_{,\underline{1}}\underline{t}_{,\underline{1}} + r_{,\underline{1}}\Lambda_{,\underline{1}}\underline{t} + (r_{,\underline{1}})^2\underline{t} + (\underline{t}_{,\underline{1}})^2 + r_{,\underline{2}}\underline{t} + \underline{t}_{,\underline{2}}r \\ &\quad + r\Lambda_{,\underline{1}}\underline{t}_{,\underline{1}} + \Lambda_{,\underline{2}}\underline{t}^2 + r\Lambda_{,\underline{2}}\underline{t}] \\ &= F(r,\Lambda,\underline{t})[C_1\underline{t}_{,\underline{2}} + C_2\underline{t}_{,\underline{1}} + (\underline{t}_{,\underline{1}})^2 + C_3\underline{t}], \text{ where} \end{aligned} \tag{54}$$

$$C_1 = \mathcal{O}(\theta_0^2 hH^*/\lambda^2), \quad C_2 = \mathcal{O}(\theta_0 H^*/\lambda^2), \quad C_3 = \mathcal{O}(\theta_0 H^*/\lambda^3).$$

We define for $N \geq 2$

$$C_{,\underline{N}} = \sum_{i=0}^{N-1} \lambda^{i+1-N} P_{,\underline{i}} + \sum_{i=0}^{N-2} \lambda^{i+2-N} Q_{,\underline{i}} \tag{55}$$

Consequently, we have from (4.52), (4.54) and the fact that $P_{,1} = \mathcal{O}(Q)$,

$$(56)\quad C_{,\underline{2}} = \sum_{i=0}^{1} \lambda^{i-1} P_{,\underline{1}} + Q = F(r,\Lambda,\underline{t})[C_1 \underline{t}_{,\underline{2}} + C_2 \underline{t}_{,\underline{1}} + (\underline{t}_{,\underline{1}})^2 + C_3 \underline{t}],$$

where $C_1 = \mathcal{O}(\theta_0^2 h H^*/\lambda^2)$, $C_2 = \mathcal{O}(\theta_0 H^*/\lambda^2)$, and $C_3 = \mathcal{O}(\theta_0 H^*/\lambda^3)$.

We investigate the non-linearity occurring in the boundary expression $B_{,\underline{2}}$. If $S^{i\pm}$ represents the surface force per unit area of the deformed surface in the Lagrange coordinate system, we have from (2.4) on the upper and lower faces of the shell

$$(57)\qquad S^{i\pm} = (t^{i3\pm} \mp h_{,\alpha} t^{i\alpha\pm})\left[x^r_{,3} \xi_r\right]^{\pm},$$

where $\xi_r^{\pm}$ are the direction cosines of the normal to the deformed faces. We have in terms of ε_{ijk} the skew symmetric ε-system tensor

$$(58)\quad \xi_r = \varepsilon_{rk\ell} x^k_{,1} x^\ell_{,2} \left[\varepsilon_{pst} x^s_{,1} x^t_{,2} \varepsilon_{\ell mn} x^m_{,1} x^n_{,2} G^{\ell p}\right]^{-1/2}$$

From (1.2a) and (4.58) we have

$$x^r_{,3} \xi_r = \left[\bar{g}^3_3\right]^{-1/2}.$$

By the tensor transformation rules of (4.1) we have

$$\hat{t}^{m3} = \frac{\partial \underline{U}^m}{\partial U^r}\left(\frac{H}{h} t^{r3} - \frac{U^3}{h}\frac{H}{h} h_{,\alpha} t^{r\alpha}\right),$$

or, multiplying by $\dfrac{\partial U^p}{\partial \underline{U}^m}\dfrac{h}{H}$, and summing

$$\frac{h}{H}\hat{t}^{m3}\frac{\partial U^p}{\partial \underline{U}^m} = t^{p3} - h_{,\alpha}\frac{\underline{U}^3}{H} t^{p\alpha}.$$

On the surfaces given by $\underline{U}^3 = \pm H$ we have, using the notation of (4.3)

$$\underline{t}^{\beta 3\pm} = (t^{\beta 3\pm} \mp h_{,\alpha} t^{\alpha\beta\pm}) = S^{\beta\pm}\left[\left(\bar{g}^3_3\right)^{1/2}\right]^{\pm},$$

$$t^{33\pm} = (t^{33\pm} \mp h_{,\alpha} t^{3\alpha\pm}) \pm h_{,\beta}(t^{\beta 3\pm} \mp h_{,\alpha} t^{\beta\alpha\pm})$$

$$= (S^{3\pm} \pm h_{,\beta} S^{\beta\pm})\left[\left(\bar{g}^3_3\right)^{1/2}\right]^{\pm}.$$

From (1.2) and (1.14) we have

$$\left(\bar{g}^3_3\right)^{1/2} = 1 + F(\eta,t)[\eta+t] = 1 + F(\eta,\Lambda,\underline{t})[\eta+\underline{t}].$$

From the expression for $S_{,\underline{2}}$ (4.43) and our assumptions about the surface forces enumerated in (2.4) we have

$$S_{,\underline{2}} = S^*_{,\underline{2}} + \mathcal{O}(\varepsilon\lambda^{-2}),$$

where by the structure of $\bar{g}^3_3$ and (2.4)

$$S^*_{,\underline{2}} = \mathcal{O}(C_{,\underline{2}}).$$

4.6 Estimates for the L_2 Norms of the First and Second Derivatives of the Stress Components $\underline{t}$

We begin by estimating $C_{,\underline{2}}$ in terms of the L_2 norms of the derivatives of the stresses. We have from (A4) and (A9) of the appendix

$$\|\phi\underline{t}^e_{,\underline{1}}\underline{t}^e_{,\underline{1}}\| = \mathcal{O}(\varepsilon[\|\phi\underline{t}^e_{,\underline{2}}\| + \lambda^{-2}\|\underline{t}\|]),$$

$$\|\phi\underline{t}_{,\underline{1}}\underline{t}_{,\underline{1}}h\| = \mathcal{O}(\varepsilon[H\|\phi\underline{t}^e_{,\underline{2}}\| + (H^*)^{-1}\|\underline{t}\|]).$$

From (4.56) and (4.60) we have, using (3.6)

$$\|\phi C_{,\underline{2}}\| = \mathcal{O}(\theta_0\lambda^{-1}\|\phi h\underline{t}_{,\underline{2}}\| + \lambda^{-2}\|\phi h\underline{t}_{,\underline{1}}\| + \lambda^{-2}\|\underline{t}\|).$$

From (4.53a) and (4.60) we have, using (3.6)

$$\lambda\|\phi h^{-1}P^e_{,\underline{1}}\| = \mathcal{O}(\theta_0\lambda^{-1}\|\phi h\underline{t}_{,\underline{2}}\| + \lambda^{-1}\|\phi\underline{t}^e_{,\underline{1}}\| + \lambda^{-2}\|\phi h\underline{t}_{,\underline{1}}\|$$
$$+ \lambda\|\phi h^{-1}\underline{t}_{,\underline{1}}t^e_{,\underline{1}}\| + \lambda^{-2}\|\underline{t}\|).$$

By Schwarz's inequality we have

$$\lambda\|\phi h^{-1}\underline{t}_{,\underline{1}}\underline{t}^e_{,\underline{1}}\| = \mathcal{O}(\|\phi\underline{t}_{,\underline{1}}\underline{t}_{,\underline{1}}\|) + \sigma(\lambda^2 H*^{-2}\|\phi\underline{t}^e_{,\underline{1}}\underline{t}^e_{,\underline{1}}\|).$$

Using (4.60) and (3.6) gives

$$\lambda\|\phi h^{-1}\underline{t}_{,\underline{1}}\underline{t}^e_{,\underline{1}}\| = \mathcal{O}(\varepsilon(\|\phi\underline{t}_{,\underline{2}}\| + H*^{-2}\|\underline{t}\|)) + \sigma(\varepsilon\lambda^2(H*)^{-2}\|\phi\underline{t}^e_{,\underline{2}}\|)$$
$$= \mathcal{O}(\theta\lambda^{-1}\|\phi h\underline{t}_{,\underline{2}}\| + \lambda^{-2}\|\underline{t}\|) + \sigma(\|\phi\underline{t}^e_{,\underline{2}}\|).$$

Hence, using (4.26) to estimate the first derivatives of $\underline{t}$,

$$[\lambda\|\phi h^{-1}P^e_{,\underline{1}}\|,\ \|\phi C_{,\underline{2}}\|] = \mathcal{O}(\lambda^{-1}\|\phi h\underline{t}_{,\underline{2}}\| + \lambda^{-2}\|\underline{t}\|$$
$$+ \theta_0\|\phi\underline{t}^e_{,\underline{2}}\|) + \sigma(\|\phi\underline{t}^e_{,\underline{2}}\|).$$

To estimate $\underline{t}^e_{,\underline{2}}$ we substitute into the estimate (4.40), the various estimates for $S_1,\ldots,S_5$ (4.44), B (4.45-47), S_6 (4.48), V (4.49), $Z^e_{,\underline{2}}$ (4.50), $S_{,\underline{2}}$ (4.59), $C_{,\underline{2}}$ and $P^e_{,\underline{1}}$ (4.61). Next, using the estimates for the first derivatives of $\underline{t}$ (4.26) we have

$$\|\phi\underline{t}^e_{,\underline{2}}\| = \mathcal{O}(\lambda^{-2}\|\underline{t}\| + \lambda^{-1}\|\phi h\underline{t}_{,\underline{2}}\| + \varepsilon\lambda^{-2}\|\phi\|$$
$$\theta_0\|\phi\underline{t}^e_{,\underline{2}}\|) + \sigma(\|\phi\underline{t}^e_{,\underline{2}}\|).$$

To estimate $\underline{t}_{,\underline{2}}$ we substitute into the estimate (4.25) and get

$$\lambda^{-1}\|\phi h\underline{t}_{,\underline{2}}\| = \mathcal{O}(\lambda^{-2}\|\underline{t}\| + \theta_0\|\phi\underline{t}^{e}_{,\underline{2}}\| + \varepsilon\lambda^{-2}\|\phi\|) + \sigma(\|\phi\underline{t}^{e}_{,\underline{2}}\|).$$

Substituting (4.63) into (4.62) for the term $\lambda^{-1}\|\phi\underline{t}_{,\underline{2}}\|$ and choosing θ_0 sufficiently small, we have

$$(\lambda^{-1}\|\phi h\underline{t}_{,\underline{2}}\|, \|\phi\underline{t}^{e}_{,\underline{2}}\|) = \mathcal{O}(\lambda^{-2}\|\underline{t}\| + \varepsilon\lambda^{-2}\|\phi\|).$$

From (4.26) we have for the norms of the first derivatives of the stresses

$$(\lambda^{-1}\|\phi h\underline{t}_{,\underline{1}}\|, \|\phi\underline{t}^{e}_{,\underline{1}}\|) = \mathcal{O}(\lambda^{-1}\|\underline{t}\| + \varepsilon\lambda^{-1}\|\phi\|),$$

substituting into (4.60) we have

$$\|\phi\underline{t}^{e}_{,\underline{1}}\underline{t}^{e}_{,\underline{1}}\| = \mathcal{O}(\varepsilon\lambda^{-2}(\varepsilon\|\phi\| + \|\underline{t}\|)),$$

$$\|\phi h(\underline{t}_{,\underline{1}})^2\| = \mathcal{O}(\varepsilon H^{*-1}(\|\underline{t}\|+\varepsilon\|\phi\|)), \quad \|\phi A_{,\underline{1}}\| = \mathcal{O}(\varepsilon\lambda^{-1}\|\phi\|+\lambda^{-1}\|\underline{t}\|).$$

Chapter 5 - Estimates for the L_2 Norms of the N-order Derivatives of the Stresses

Our procedure in estimating higher order derivatives of the stresses hinges on the fact that much of the second order analysis can be adapted. We find that $\underline{t}^{ik}_{,\underline{\beta}}$ for fixed $\underline{\beta}$ will satisfy basically the same set of equations as the $\underline{t}^{ik}$ did, provided we replace $P, Q, Z_{,\underline{2}}V, \ldots$ by $P^{(0)}_{,\underline{1}}, Q^{(0)}_{,\underline{1}}, Z_{,\underline{3}}, V^{(0)}_{,\underline{1}}, \ldots$. The equation of compatibility can be used to estimate any derivative of even order with respect to $\underline{U}^3$ by a combination of derivatives with respect to $\underline{U}^1$ and $\underline{U}^2$. In addition, we will find that differentiation with respect to $\underline{U}^\alpha$ of the stresses or their derivatives of any order introduces a factor of λ^{-1} in our estimates.

As a beginning, we recall the definitions of the quantities $\underline{t}^{(\mu)(s)}_{,\underline{N}}$ (4.9), $Z^{(\mu,s)}_{,\underline{N}}$ (4.12), $A_{,\underline{N}}$ (4.17), $S_{,\underline{N}}$ (4.43), $B_{,\underline{N}}$ (4.45), and $C_{,\underline{N}}$ (4.55). For the quantity $\underline{t}^{(\mu)(s)}_{,\underline{N}}$, $N > 2$ we can consider three cases (i) $\mu - s < 0$ and odd, (ii) $\mu - s > 0$, (iii) $\mu - s \leq 0$ and even. For case (i) $\mu - s < 0$ and odd, we have from the equilibrium equations in the U coordinate system (1.16), transforming to $\underline{U}$ coordinate system and introducing $Z_{,\underline{N}}$

$$(1) \quad \lambda^{-1}\|\phi h\underline{t}^{(\mu)(s)}_{,\underline{N}}\| = \mathcal{O}(\lambda^{-1}\|\phi h\underline{t}^{(0)(s-\mu)}_{,\underline{N}}\| + \lambda^{-1}\|\phi h C_{,\underline{N}}\| + \lambda^{-1}\|\phi h Z_{,\underline{N}}\| + \lambda^{-1}\|\phi h B_{,\underline{N-1}}\|).$$

By (A8) of the appendix we have, provided $(s-\mu) < N$

$$(2)\quad \lambda^{-1}\|\phi h\underline{t}^{(0)(s-\mu)}_{,\underline{N}}\| = \mathcal{O}(H\lambda^{-1}(\|\phi\underline{t}^{(0)(s-\mu-1)}_{,\underline{N}}\| + \|\phi\underline{t}^{(0)(s-\mu+1)}_{,\underline{N}}\|)$$

$$+ \lambda^{-1}\|\phi hZ_{,\underline{N}}\| + \lambda^{-2}\|\underline{t}^{(0)(s-\mu-1)}_{,\underline{N-2}}\|) + \sigma(\|\phi\underline{t}^{(0)(s-\mu-1)}_{,\underline{N}}\|).$$

If $s-\mu = N$ (>2) we employ the compatibility equation in the same way as was done in (4.22-23) to reduce $(s-\mu)$ by 2. That is, for general $r > 2$ we have

$$(3)\quad \lambda^{-1}\|\phi h\underline{t}^{(0)(r)}_{,\underline{N}}\| = \mathcal{O}(\lambda^{-1}\|\phi h\underline{t}^{(0)(r-2)}_{,\underline{N}}\| + \lambda^{-1}\|\phi h\underline{t}^{(2)(r-2)}_{,\underline{N}}\|$$

$$+ \lambda^{-1}\|\phi hC_{,\underline{N}}\| + \lambda^{-1}\|\phi hZ_{,\underline{N}}\| + \lambda^{-1}\|\phi hB_{,\underline{N-1}}\|).$$

Combining the estimates (5.1-3) we have for $\mu-s < 0$ and odd

$$\lambda^{-1}\|\phi h\underline{t}^{(\mu)(s)}_{,\underline{N}}\| = \mathcal{O}(\lambda H^{-1}\|\phi\underline{t}^{(0)(r-1)}_{,\underline{N}}\| + \|\phi\underline{t}^{(0)(r+1)}_{,\underline{N}}\|$$

$$+ \lambda^{-1}\|\phi hZ_{,\underline{N}}\| + \lambda^{-2}\|\underline{t}^{(0)(r-1)}_{,\underline{N-2}}\| + \lambda^{-1}\|\phi C_{,\underline{N}}\|$$

$$+ \lambda^{-1}\|\phi hB_{,\underline{N-1}}\|) + \sigma(\|\phi\underline{t}^{(0)(r-1)}_{,\underline{N}}\|),\quad \text{for some } r \text{ odd.}$$

where $0 < r < N$, $N > 2$.

For case (ii) if $(\mu-s) > 0$,

$$(4)\qquad \|\phi\underline{t}^{(\mu)(s)}_{,\underline{N}}\| = \mathcal{O}(\|\phi A_{,\underline{N}}\|).$$

For case (iii) if $(\mu-s) \leq 0$ and even, we have from the equilibrium equations

$$\|\phi\underline{t}^{(\mu)(s)}_{,\underline{N}}\| = \mathcal{O}(\|\phi\underline{t}^{(0)(s-\mu)}_{,\underline{N}}\| + \|\phi C_{,\underline{N}}\|$$

$$+ \sum_{r=1}^{\mu}\left[\|\phi Z^{(\mu-r,s-r)}_{,\underline{N}}\| + \|\phi B^{(\mu-r)(s-r)}_{,\underline{N-1}}\|\right]).$$

By the compatibility equation, we have for $s-\mu \geq 2$

$$\|\phi\underline{t}^{(0)(s-\mu)}_{,\underline{N}}\| = \mathcal{O}(\|\phi\underline{t}^{(0)(s-\mu-2)}_{,\underline{N}}\| + \|\phi\underline{t}^{(2)(s-\mu-2)}_{,\underline{N}}\|$$
$$+ \|\phi B^{(1)(s-\mu-1)}_{,\underline{N-1}}\| + \|\phi B^{(0)(s-\mu-2)}_{,\underline{N-1}}\| + \|\phi C_{,\underline{N}}\| + \|\phi Z^{e}_{,\underline{N}}\|).$$

Continuing in this fashion to estimate $\underline{t}^{(0)(p)}_{,\underline{N}}$ as long as $p > 2$, we eventually arrive at the estimate

$$\|\phi\underline{t}^{e}_{,\underline{N}}\| = \mathcal{O}(\|\phi C_{,\underline{N}}\| + \|\phi A_{,\underline{N}}\| + \|\phi\underline{t}^{(0)(0)}_{,\underline{N}}\|$$
$$+ (\|\phi B^{e}_{,\underline{N-1}}\| + \|\phi Z^{e}_{,\underline{N}}\|)).$$

If we differentiate the equations involved in the estimates of $\underline{t}^{(0)(0)}_{,\underline{2}}$ by $U^{\alpha_1},\ldots,U^{\alpha_{N-2}}$ (see 4.14-23), we have

$$\|\phi\underline{t}^{(0)(0)}_{,\underline{N}}\| = \mathcal{O}(\|\phi A_{,\underline{N}}\| + \|\phi C_{,\underline{N}}\| + \lambda^{-2}\|\underline{t}^{(0)(0)}_{,\underline{N-2}}\|$$
$$+ \|\phi(Z^{(0,0)}_{,\underline{N}} + Z^{(2,0)}_{,\underline{N}} + Z^{(0,2)}_{,\underline{N}} + Z^{(2,2)}_{,\underline{N}})\|$$
$$+ \|\phi B^{(1)(1)}_{,\underline{N-1}}\| + \|\phi B^{(0)(0)}_{,\underline{N-1}}\|).$$

Consequently,

$$(5)\qquad \|\phi\underline{t}^{e}_{,\underline{N}}\| = \mathcal{O}\Big(\|\phi(C_{,\underline{N}} + A_{,\underline{N}})\| + \sum_{\substack{\mu-s\geq 0\\ \text{or even}}}^{\mu}\Big(\sum_{r=1}^{\mu}\|\phi B^{e}_{,\underline{N-1}}\|$$
$$+ \|\phi Z^{e}_{,\underline{N}}\|\Big) + \lambda^{-2}\|\underline{t}^{e}_{,\underline{N-2}}\|\Big).$$

$$(6)\qquad \lambda^{-1}\|\phi h\underline{t}_{,\underline{N}}\| = \mathcal{O}(\theta_0(\|\phi\underline{t}^{e}_{,\underline{N}}\| + \lambda^{-2}\|\underline{t}^{e}_{,\underline{N-2}}\|)$$
$$+ \lambda^{-1}\|\phi h(C_{,\underline{N}} + Z_{,\underline{N}} + B_{,\underline{N-1}})\|) + \sigma(\|\phi\underline{t}^{e}_{,\underline{N}}\|).$$

In estimating $A_{,\underline{N}}$ we follow the procedure used in estimating $A_{,\underline{2}}$. The most important ingredient is the compatibility equation (4.8) and the equilibrium equations (4.33). Integrating by parts numerous times, we obtain:

$$\|\phi A_{,\underline{N}}\| = \mathcal{O}\Big(\sum_{i=1}^{N}\Big[\lambda^{-i}\big[\|A_{,\underline{N-i}}\| + \|\underline{t}^{e}_{,\underline{N-i}}\| + \|\phi\lambda^{-1}h\underline{t}_{,\underline{N-i}}\|\big]$$

$$+ \lambda^{-i+1}\|\Lambda B_{,\underline{N-i}}\| + \lambda^{-i+1}\|B^{e}_{,\underline{N-i}}\|\Big] + \|\phi S_{,\underline{N}}\|$$

$$+ (\sup\,\Lambda)\|\phi A_{,\underline{N}}\| + \|\phi C_{,\underline{N}}\| + {}^{-1}\|\phi h\underline{t}_{,\underline{N}}\|\Big)$$

$$+ \sigma\Big(\sum_{i=1}^{N}\lambda^{2-i}\|\phi h^{-1}V^{(0)}_{,\underline{N-i}}\| + \|\phi A_{,\underline{N}}\| + \|\phi\underline{t}^{e}_{,\underline{N}}\|\Big),$$

where $\sup\,\Lambda = \ (\theta_0)$ by (3.7).

Substituting the estimate for $S_{,\underline{N}}$ and for the body forces from (4.43) and (4.46), and using the method of (4.49) to construct an estimate for $V_{,\underline{N-i}}$, we have by (5.4) and (5.6),

(7)
$$\|\phi A_{,\underline{N}}\| = \mathcal{O}\Big(\sum_{i=1}^{N}\lambda^{-i}\big[\|\underline{t}^{e}_{,\underline{N-i}}\| + \lambda^{-1}\|\phi hZ_{,\underline{N-i}}\|\big]$$

$$+ \theta_0\|\phi\underline{t}^{e}_{,\underline{N}}\| + \varepsilon\lambda^{-N}\|\phi\| + \|\phi C_{,\underline{N}}\| + \lambda^{-1}\|\phi hZ_{,\underline{N}}\|\Big)$$

$$+ \sigma\Big(\|\phi\underline{t}^{e}_{,\underline{N}}\| + \sum_{i=1}^{N}\lambda^{2-i}\|\phi h^{-1}P^{(0)}_{,\underline{N-i}}\|\Big).$$

Altogether, we have from (5.5 , 5.7) and after expressing $Z_{,\underline{N}}$ in terms of the stresses,

(8)
$$\|\phi\underline{t}^{e}_{,\underline{N}}\| = \mathcal{O}\Big(\|\phi C_{,\underline{N}}\| + \Big(\sum_{i=1}^{N}\lambda^{-i}\|\underline{t}^{e}_{,\underline{N-i}}\|$$

$$+ \theta_0\|\phi\underline{t}^{e}_{,\underline{N}}\|\Big) + \varepsilon\lambda^{-N}\|\phi\|\Big) + \sigma\Big(\sum_{i=1}^{N}\lambda^{2-i}\|\phi h^{-1}P^{(0)}_{,\underline{N-i}}\|\Big)$$

(9)
$$\lambda^{-1}\|\phi h\underline{t}_{,\underline{N}}\| = \mathcal{O}\Big(\theta_0\big[\|\phi C_{,\underline{N}}\| + \sum_{i=0}^{N}\lambda^{-i}\|\underline{t}^{e}_{,\underline{N-i}}\|\big]$$

$$+ \varepsilon\lambda^{-N}\|\phi\|\Big) + \sigma(\|\phi\underline{t}^{e}_{,\underline{N}}\|).$$

In the preceding estimates we have always used the function ϕ without regard to which of the set $\{\phi_i\}$ we have chosen. Now we use the notation

$$\|\phi_i f\|_i = \iiint\limits_{\underline{M}_i} (\phi_i f)^2 d\underline{U}^1 d\underline{U}^2 d\underline{U}^3,$$

$$\underline{M}_i = \begin{cases} \{(\underline{U}^1,\underline{U}^2,\underline{U}^3) \mid \underline{U}^\alpha \underline{U}^\alpha < \lambda^2, |\underline{U}^3| < H\}, & i = 0,1,2. \\ \{(\underline{U}^1,\underline{U}^2,\underline{U}^3) \mid \underline{U}^\alpha \underline{U}^\alpha < \frac{\lambda^2}{4^{i-2}}, |\underline{U}^3| < H\}, & i = 3,4,\ldots. \end{cases}$$

We observe that $1 = \mathcal{O}(\phi_i)$ in region $\underline{M}_j$ $j < i$, and hence $\|w\|_i = \mathcal{O}(\|\phi_j w\|_j)$.

We also employ the convention that if no index appears on the norm sign the region of integration is given by the function ϕ_i, e.g., $\|\phi_i\| = \|\phi_i\|_i$, and for example from (5.9)

$$\|\phi_N \underline{t}_{,\underline{N}}\| = \mathcal{O}(\theta_0[\|\phi_N C_{,\underline{N}}\| + \sum_{i=0}^{N} \lambda^{-i} \|\phi_{N-i} \underline{t}^e_{,\underline{N-i}}\|]$$
$$+ \varepsilon\lambda^{-N}\|\phi_N\|) + \sigma(\|\phi \underline{t}^e_{,\underline{N}}\|).$$

It now remains to estimate the terms, such as $C_{,\underline{N}}$, containing the non-linear behavior of the equations of equilibrium and compatibility, and the boundary conditions.

Generally, we have that $P_{,\underline{N}}$ and $Q_{,\underline{N-1}}$ have the same form as an expression in the quantities $(\Lambda, \eta, \underline{t})$ and their derivatives. The same is true for $\lambda^{-r} P_{,\underline{s}}$ and $P_{,\underline{r+s}}$ and similarly for $\lambda^{-r} Q_{,\underline{s}}$ and $Q_{,\underline{r+s}}$. From (4.55) we calculate $C_{,\underline{3}}$ using (4.52) and (4.53). We find that

$$C_{,\underline{3}} = F(\Lambda, \eta, \underline{t})[C_1 \underline{t}_{,\underline{3}} + \underline{t}_{,\underline{1}} \underline{t}_{,\underline{3}} + (\underline{t}_{,\underline{1}})^3 + C_2 (\underline{t}_{,\underline{1}})^2$$
$$+ C_3 \underline{t}_{,\underline{2}} + C_4 \underline{t}_{,\underline{1}} + C_5 \underline{t}], \quad \text{where}$$

$C_1 = \mathcal{O}(\theta_0^2 h H^*/\lambda^2)$, $C_2 = \mathcal{O}(1/\lambda)$, $C_3 = \mathcal{O}(\theta_0 h/\lambda^2)$,

$C_4 = \mathcal{O}(\theta_0 h/\lambda^3)$ and $C_5 = \mathcal{O}(\theta_0 h/\lambda^4)$.

Noting that $H < H(H^*)^{-1} h(U^1, U^2)$, we have

$$(10) \quad H\|\phi_3 C_{,3}\| = \mathcal{O}\big(\theta_0 \|\phi_3 h \underline{t}_{,\underline{3}}\| + H(H^*)^{-1}(\|\phi_3 h \underline{t}_{,\underline{1}} \underline{t}_{,\underline{2}}\| + \|\phi_3 h(\underline{t}_{,\underline{1}})^3\| + \lambda^{-1}\|\phi_3 h(\underline{t}_{,\underline{1}})^2\|) + \theta_0 \lambda^{-1}\|\phi_3 h \underline{t}_{,2}\| + \theta_0 \lambda^{-2}\|\phi_3 h \underline{t}_{,\underline{1}}\| + \theta\lambda^{-3}\|\phi h \underline{t}\|\big).$$

It is necessary in estimating $C_{,\underline{3}}$ to employ a notation for the pointwise norms of the derivatives of the stresses. We then make use of a Sobolev type inequality in estimating the non-linear terms. For this reason we introduce T_1 and $T_1^{(0)}$ as follows:

$$(11 \quad T_1 = \max_{\underline{M}_3} |\phi_3 \underline{t}_{,\underline{1}} h|, \quad T_1^{(0)} = \max_{\underline{M}_3} |\phi_3 \underline{t}_{,\underline{1}}^{(0)}|,$$

and for the N-order derivative

$$(11b) \quad T_N = \max_{\underline{M}_{N+2}} |\phi_{N+2} \underline{t}_{,\underline{N}} h|, \quad T_N^{(0)} = \max_{\underline{M}_{N+2}} |\phi_{N+2} \underline{t}_{,\underline{N}}^{(0)}|.$$

Employing this notation, we write using the estimate (4.69) and the inequality (3.6),

$$(12) \quad \frac{H}{H^*}\|\phi_3 h \underline{t}_{,\underline{1}} \underline{t}_{,\underline{2}}\| \leq T_1 \frac{H}{(H^*)^2}\|h \underline{t}_{,\underline{2}}\|_3 \leq T_1 \frac{H}{(H^*)^2}\|\phi_1 h \underline{t}_{,\underline{2}}\|$$
$$= \mathcal{O}(\theta_0 T_1 \sqrt{H}\lambda^{-1}), \quad \text{where} \quad \|1\|_i = \mathcal{O}(\sqrt{H}\lambda).$$

Using Sobolev's lemma (A10) of the appendix, gives

$$(13) \quad T_1 \sqrt{H}\lambda^{-1} = \mathcal{O}(\|\phi_3 \underline{t}_{,\underline{3}} h\| + \lambda^{-2}\|\underline{t}\|_0).$$

Similarly, we have

$$(14)\quad \frac{H}{H^*}\|\phi_3 h(\underline{t}_{,\underline{1}})^3\| \le \frac{H}{(H^*)^2}T_1\|h(\underline{t}_{,\underline{1}})^2\|_3 = \mathcal{O}\Big((\varepsilon)^2\frac{\cdot H}{(H^*)^2}T_1\frac{1}{H^*}\lambda\sqrt{H}\Big).$$
$$= \mathcal{O}(T_1\sqrt{H}\lambda^{-1}) = \mathcal{O}(\|\phi_3\underline{t}_{,\underline{3}}h\| + \lambda^{-2}\|\underline{t}\|_0).$$

Estimating the remaining terms in (5.10) in a similar fashion gives

$$(15)\quad H\|\phi_3 C_{,\underline{3}}\| = \mathcal{O}(\theta\|\phi_3 h\underline{t}_{,\underline{3}}\| + \theta_0\lambda^{-2}\|\underline{t}\|_0) + \sigma(\|\phi_3\underline{t}^e_{,\underline{3}}\|).$$

To estimate the L_2 norms

$$\|\phi_3 h^{-1}P^{(0)}_{,\underline{2}}\| \quad\text{and}\quad \|\phi_3 h^{-1}P^{(0)}_{,\underline{1}}\|$$

we calculate the form of the expression $P^{(0)}_{,\underline{2}}$. From (4.52) we have

$$(16)\quad P^{(0)}_{,\underline{2}} = F(\eta,\Lambda,\underline{t})[\underline{t}^{(0)}_{,\underline{1}}\underline{t}_{,\underline{2}} + \underline{t}_{,\underline{1}}\underline{t}^{(0)}_{,\underline{2}} + \big[\underline{t}^{(0)}_{,\underline{1}}\big]^2\underline{t}_{,\underline{1}}$$
$$+ C_1\underline{t}_{,\underline{1}}\underline{t}^{(0)}_{,\underline{1}} + C_2\underline{t}^{(0)}_{,\underline{1}}\underline{t}^{(0)}_{,\underline{1}} + C_3\underline{t}_{,\underline{2}} + C_4\underline{t}^{(0)}_{,\underline{2}}$$
$$+ C_5\underline{t}_{,\underline{3}} + C_6\underline{t}^{(0)}_{,\underline{1}} + C_7\underline{t}_{,1} + C_8\underline{t}],\quad\text{where}$$

$$C_1 = \mathcal{O}(\theta_0 H^*/\lambda^2),\quad C_2 = \mathcal{O}(1/\lambda),\quad C_3 = \mathcal{O}(\theta_0^2 H^* h/\lambda^3),$$
$$C_4 = \mathcal{O}(\theta_0 H^*/\lambda^2),\quad C_5 = \mathcal{O}(\theta_0^2 H^* h/\lambda),\quad C_6 = \mathcal{O}(\theta_0 H^*/\lambda^3),$$
$$C_7 = \mathcal{O}(\theta_0^2 H^* h/\lambda^4),\quad C_8 = \mathcal{O}(\theta_0 H^*/\lambda^4).$$

We note from (4.54a) that this is also the form of $\lambda^{-1}P^{(0)}_{,\underline{1}}$. Consequently, using (5.11-5.14) to estimate the terms in (5.16) gives

$$\|\phi_3 h^{-1} p^{(0)}_{,\underline{2}}\| + \lambda^{-1}\|\phi_3 h^{-1} p^{(0)}_{,\underline{1}}\| = \mathcal{O}\Big(\Big[\frac{1}{H^*}\Big]^2\Big[\|h\underline{t}\alpha_{,\underline{2}}\|_3 T_1^{(0)}$$
$$+ T_1\|\underline{t}^{(0)}_{,\underline{2}}\|_3 + \lambda^{-1}\|\underline{t}^{(0)}_{,\underline{1}}\|_3 + T_1\|(\underline{t}^{(0)}_{,\underline{1}})^2\|_3\Big]$$
$$+ \frac{1}{H^*\lambda}\|\phi_3 \underline{t}^{(0)}_{,1}\underline{t}^{(0)}_{,\underline{1}}\| + \lambda^{-3}\|\phi_3 h\underline{t}_{,\underline{2}}\| + \lambda^{-2}\|\phi_3 \underline{t}^{(0)}_{,\underline{2}}\|$$
$$+ \lambda^{-2}\|\phi_3 h\underline{t}_{,\underline{3}}\| + \lambda^{-3}\|\phi_3 \underline{t}^{(0)}_{,\underline{1}}\| + \lambda^{-4}\|\phi_3 h\underline{t}_{,1}\|$$
$$+ \lambda^{-4}\|\phi_3\underline{t}\|\Big).$$

Using the previous results for $N \leq 2$ (4.64-65) we have

$$(17)\quad \|\phi_3 h^{-1} p^{(0)}_{,\underline{2}}\| + \lambda^{-1}\|\phi_3 h^{-1} p^{(0)}_{,\underline{1}}\| = \mathcal{O}\Big(T_1^{(0)}\lambda^{-3}(\sqrt{H}\lambda)$$
$$+ T_1\lambda^{-4}(\sqrt{H}\lambda) + \varepsilon\lambda^{-4}(\sqrt{H}\lambda) + \lambda^{-2}\|\phi_3 h\underline{t}_{,\underline{3}}\|\Big).$$

Using the formula (A10) from the appendix we have

$$(18)\quad T_1^{(0)}\sqrt{H}\lambda = \mathcal{O}\Big(\lambda^2\|\phi_3\underline{t}^{(0)}_{,\underline{3}}\| + H\|\phi_3 h\underline{t}_{,\underline{3}}\| + \lambda^{-1}\|\underline{t}\|_0\Big).$$

Substituting (5.13) and (5.18) into (5.17) gives

$$(19)\quad \|\phi_3 h^{-1} p^{(0)}_{,\underline{2}}\| + \lambda^{-1}\|\phi_3 h^{-1} p^{(0)}_{,\underline{1}}\| = \mathcal{O}\Big(\lambda^{-1}\|\phi_3\underline{t}^{(0)}_{,\underline{3}}\|$$
$$+ \lambda^{-2}\|\phi_3 h\underline{t}_{,\underline{3}}\| + \lambda^{-4}\|\underline{t}\|_0 + \varepsilon\lambda^{-4}(\sqrt{H}\lambda)\Big).$$

From (5.8) we have, upon substituting the estimates (5.15), (5.19), (4.64), (4.65),

$$(20a)\quad \|\phi_3\underline{t}^e_{,\underline{3}}\| = \mathcal{O}\Big(\lambda^{-1}\|\phi_3 h\underline{t}_{,\underline{3}}\| + \varepsilon\lambda^{-3}(\sqrt{H}\lambda)$$
$$+ \lambda^{-3}\|\underline{t}\|_0 + \theta_0\|\phi_3\underline{t}^e_{,\underline{3}}\|\Big) + o(\|\phi_3\underline{t}^e_{,\underline{3}}\|).$$

and from (5.9)

$$(20b)\quad \lambda^{-1}\|\phi_3 h\underline{t}_{,\underline{3}}\| = \mathcal{O}\Big(\theta_0\big[\lambda^{-1}\|\phi_3 h\underline{t}_{,\underline{3}}\| + \|\phi_3 \underline{t}^e_{,\underline{3}}\|\big] + \varepsilon\lambda^{-3}(\sqrt{H}\lambda) + \lambda^{-3}\|\underline{t}\|_0\Big) + o(\|\phi_3 \underline{t}^e_{,\underline{3}}\|).$$

If θ_0 is sufficiently small, we have from (5.20)

$$(21a)\quad \|\phi_3 \underline{t}^3_{,\underline{3}}\| = \mathcal{O}\Big(\varepsilon\lambda^{-3}(\sqrt{H}\lambda) + \lambda^{-3}\|\underline{t}\|_0\Big) = \mathcal{O}(\varepsilon\lambda^{-3}\sqrt{H}\lambda),$$

$$(21b)\quad \lambda^{-1}\|\phi_3 h\underline{t}_{,\underline{3}}\| = \mathcal{O}(\varepsilon\lambda^{-3}(\sqrt{H}\lambda) + \lambda^{-3}\|\underline{t}\|_0) = \mathcal{O}(\varepsilon\lambda^{-3}\sqrt{H}\lambda).$$

Substituting the estimates (5.21) into (5.13) and (5.18) we have

$$|\phi_3 h\underline{t}_{,\underline{1}}| \leq T_1 = \mathcal{O}(\varepsilon),$$
$$|\phi_3 \underline{t}^{(0)}_{,\underline{1}}| \leq T_1^{(0)} = \mathcal{O}(\varepsilon\lambda^{-1}).$$

Continuing this same process by estimating $C_{,\underline{i}}$, $P^{(0)}_{,\underline{i-1}}$ and using a function ϕ_i with support contained inside the region $\underline{M}_{i-1}$, we will get pointwise estimates on T_{i-2} and $T^{(0)}_{i-2}$. $(i = 4,5,\ldots)$. Hence by choosing θ_0 sufficiently small we will be able to get pointwise estimates for as high a derivative of the stresses as we require. At the point P_0 on the midsurface we have $\underline{U}^1 = \underline{U}^2 = 0$ and $\phi_i = 1$. Consequently at P_0

$$h\underline{t}^{k_1k_2}{}_{,\underline{i_1\ldots i_N}} = \mathcal{O}(\varepsilon\lambda^{1-N}),$$

$$\underline{t}^{k_1k_2}{}_{,\underline{\alpha_1\ldots\alpha_N}} = \mathcal{O}(\varepsilon\lambda^{-N}),$$

for as high a value of N as we carry out the estimates.

By the transformation formulas (4.3,4.5) and the estimate

$\sqrt{h_{,\alpha}h_{,\alpha}} = \mathcal{O}(\theta_0(h/\lambda))$ we have from (5.22)

$$ht^{k_1k_2}{}_{,i_1\ldots i_N} = \mathcal{O}(\varepsilon\lambda^{1-N}),$$

$$t^{k_1k_2}{}_{,\alpha_1\ldots\alpha_N} = \mathcal{O}(\varepsilon\lambda^{-N}).$$

For the non-linear terms, P,Q and their derivatives, we have

$$P = \mathcal{O}(\varepsilon\lambda^{-1}\Omega), \quad P^{(0)}_{,1} = \mathcal{O}(\varepsilon\lambda^{-2}\Omega), \ldots.$$

$$P_{,1} = \mathcal{O}(\varepsilon\lambda^{-2}), \quad P_{,2} = \mathcal{O}(\varepsilon\lambda^{-3}), \ldots,$$

$$Q = \mathcal{O}(\varepsilon\lambda^{-2}), \quad Q_{,1} = \mathcal{O}(\varepsilon\lambda^{-3}), \ldots.$$

Chapter 6 - Summary of the Estimates for the Derivatives of the Stresses in a Thin Shell

From the differential equations (1.16-17) we have by an analysis similar to that carried out in (4.17-4.24)

$$t^{(\mu)(s)}_{,N} = \begin{cases} \mathcal{O}(\varepsilon\lambda^{-N}) & \mu\text{-}s \geq 0 \text{ or even,} \\ \mathcal{O}(\varepsilon\lambda^{-N+1}h^{-1}) & \mu\text{-}s < 0 \text{ and odd.} \end{cases}$$

By employing the boundary conditions we can improve the estimates on certain of the stresses and their derivatives. Since,

$$t^{k3}(U^3) = t^{k3-} + \int_{-h}^{U^3} t^{k3}{}_{,3}\, dU^3, \quad k = 1,2,3$$

we have

$$t^{\beta 3} = \mathcal{O}(\varepsilon\Omega), \ldots, t^{\beta 3}{}_{,\alpha_1\ldots\alpha_N} = \mathcal{O}(\varepsilon\Omega\lambda^{-N}),$$

$$t^{33} - \tfrac{1}{2}(t^{33+} + t^{33-}) = \mathcal{O}(\varepsilon\Omega^2),$$

$$t^{33}{}_{,\alpha_1\ldots\alpha_N} - \tfrac{1}{2}(t^{33+} + t^{33-})_{,\alpha_1\ldots\alpha_N} = \mathcal{O}(\varepsilon\Omega^2\lambda^{-N}),$$

for as high an N as we have carried out our pointwise estimates.

We summarize our result by

$$(1) \quad t^{(\mu)(s)}_{,N} - \delta^{\mu}_{2}\, \tfrac{1}{2}(t^{(2)+} + t^{(2)-})^{(s)}_{,N} = \begin{cases} \mathcal{O}(\varepsilon\lambda^{-N}) & \mu\text{-}s \leq 0 \text{ and even,} \\ \mathcal{O}(\varepsilon\lambda^{-N}\Omega^{\mu-s}) & \mu\text{-}s > 0, \\ \mathcal{O}(\varepsilon\lambda^{-N+1}h^{-1}) & \mu\text{-}s < 0, \text{ and odd} \end{cases}$$

where $\mu = 0,1,2$; $s = 0,1,\ldots,N$.

These estimates hold at a point P of the shell which lies on the normal to the unstrained midsurface Σ_0 at a point P_0, in the parametric coordinate system U^i which has been identified with the unstrained Cartesian coordinate system X^α, which has its origin at P_0 and X^3 axis normal to Σ_0 at P_0.

We can generalize the estimates (6.1) to the covariant and mixed components of t by noting that

$$t_{ik} = t^{ik} + F(\eta,t)[\eta t] = t^i_k + F(\eta,t)[\eta t],$$

$$\text{and} \quad \eta_{,N} = \mathcal{O}((\theta_0\Omega)^{-N}).$$

Consequently, it makes no difference in the estimate (6.1) whether the stresses are given in contravariant, covariant, or mixed components.

By using (3.9) and an inductive argument on the order of differentiation, we can show that

$$t^{k_1k_2}{}_{;i_1\cdots i_N} = \mathcal{O}\left(t^{(u)(s)}_{;N}\right) = \mathcal{O}\left(t^{(u)(s)}_{,N}\right)$$

Estimates valid in a particular coordinate system are valid in all coordinate systems once they are written in an invariant notation. We shall always assume that whatever coordinate system is used, U^α will be constant along normals (here given by $-h \leq U^3 \leq h$) to the undeformed surface. Under these conditions, any coordinate system U^α can be used to specify estimates. For instance, if w_{ik} denotes a covariant tensor, then w behaves as a tensor under any transformation from one (U^1,U^2) system, to another. $w_{3\alpha}$ behaves as a vector, and w_{33} behaves as a scalar. For the stresses we have $(t^{\alpha\beta}t_{\alpha\beta})^{1/2} = \mathcal{O}(\varepsilon)$, $(t_{\alpha 3}t^{3\alpha})^{1/2} = \mathcal{O}(\varepsilon\Omega)$,.... In addition, we see that any coordinate system U^1, U^2 can be picked to specify the conditions on the body and surface forces.

Finally, we remark that the Kirchhoff hypothesis that "normals to the middle surface in the unstrained state go over into curves normal to the deformed middle surface" is approximately true in the sense that the angles made by the lines $dU^\alpha = 0$ with the surface $dU^3 = 0$ are at most of order $\varepsilon\Omega$, rather than of order ε. This is seen by noting that the cosine of this angle is given by

$$\begin{aligned}\left[\overline{g}^3_3 g^3_3\right]^{1/2} &= 1 - 2\varepsilon^3_\beta \varepsilon^\beta_3 + \mathcal{O}(\varepsilon^3) \\ &= 1 - 2(1+\nu)t^3_\beta t^\beta_3 + \mathcal{O}(\varepsilon^3) \\ &= 1 + \mathcal{O}(\varepsilon^2\Omega^2).\end{aligned}$$

Chapter 7 - Description of the Stress System and Geometry of the Deformed Shell

We choose as the functions for the approximate description of the stress system in the shell the symmetric quantities

$$(1) \qquad \tau_{\alpha\beta} = (t_{\alpha\beta})_{u=0}, \quad \sigma_{\alpha\beta} = (t_{\alpha\beta,3})_{u=0},$$

where we write u for U^3. By means of the estimate (6.1) we have that

$$t_{\alpha\beta} = \tau_{\alpha\beta} + \sigma_{\alpha\beta}u + \mathcal{O}(\varepsilon\Omega^2), \quad \tau_{\alpha\beta} = \mathcal{O}(\varepsilon), \quad h\sigma_{\alpha\beta} = \mathcal{O}(\varepsilon).$$

This expresses the fact that along a normal fiber of the shell the longitudinal stress varies linearly and is determined by $\tau_{\alpha\beta}$ and $\sigma_{\alpha\beta}$ within an order of error of magnitude $\varepsilon\Omega^2$ at most. $\tau_{\alpha\beta}$ can be associated with the resultant stress on a fiber coinciding with a normal to Σ_0 in the unstrained state, and $\sigma_{\alpha\beta}$ with the moment in a similar fiber:

$$\int_{-h}^{h} t_{\alpha\beta}du = 2h\tau_{\alpha\beta} + \mathcal{O}(\varepsilon h\Omega^2),$$

$$\int_{-h}^{h} t_{\alpha\beta}udu = \frac{2}{3}h^3\sigma_{\alpha\beta} + \mathcal{O}(\varepsilon h^2\Omega^2).$$

To describe the geometry of the deformed shell, we use the coefficients $e_{\alpha\beta}$ and $\ell_{\alpha\beta}$ of the first and second fundamental forms of the deformed midsurface referred to the parametric system (U^1,U^2).

$$(2) \qquad e_{\alpha\beta} = (g_{\alpha\beta})_{u=0}$$

$$(3)\quad \ell_{\alpha\beta} = \left(\left[\bar{g}^3_3\right]^{-1/2}\bar{\Gamma}^3_{\alpha\beta}\right)_{u=0} = \left[\left[\bar{g}^3_3\right]^{-1/2}\left(L_{\alpha\beta} + C^3_{\alpha\beta}\right)\right]_{u=0}.$$

We refer all tensor operations performed on the surface Σ_0 to the metric defined by

$$dS^2 = E_{\alpha\beta}dU^\alpha dU^\beta.$$

By (2.1) we have on the surface Σ_0:

$$(G_{\alpha\beta})_{u=0} = E_{\alpha\beta}, \quad (G^{\alpha\beta})_{u=0} = E^{\alpha\beta},$$

$$G^{\alpha 3} = G_{\alpha 3} = 0, \quad G^{33} = G_{33} = 1.$$

On this surface, passage from the covariant to contravariant components is the same as that defined for a two-dimensional tensor. For example, the tensor w^i_{kr} gives rise to the two-dimensional tensors $w^\alpha_{\beta\gamma}, w^3_{\beta\gamma}, w^\alpha_{3\gamma}, w^\alpha_{\beta 3}, \ldots,$ on the surface Σ_0.

Covariant differentiation with respect to the metric $dS^2 = E_{\alpha\beta}dU^\alpha dU^\beta$ will be defined by the symbol "$|$" and we will use the Christoffel symbols

$$\left\{ {\alpha \atop \beta\gamma} \right\} = \tfrac{1}{2}E^{\alpha\mu}(E_{\beta\mu,\gamma} + E_{\gamma\mu,\beta} - E_{\beta\gamma,\mu}) = \{\Gamma^\alpha_{\beta\gamma}\}_{u=0}.$$

Covariant differentiation in the $ds^2 = e_{\alpha\beta}dU^\alpha dU^\beta$ metric will be defined by the symbol "$\|$" and will employ the Christoffel symbols

$$(4)\quad \overline{\left\{ {\alpha \atop \beta\gamma} \right\}} = \left\{ {\alpha \atop \beta\gamma} \right\} + c^\alpha_{\beta\gamma}, \quad c^\alpha_{\beta\gamma} = \tfrac{1}{2}\bar{e}^{\alpha\mu}(e_{\mu\beta|\gamma} + e_{\mu\gamma|\beta} - e_{\beta\gamma|\mu}),$$

where $c^\alpha_{\beta\gamma}$ is a tensor, and $\bar{e}^{\alpha\mu}e_{\mu\beta} = \delta^\alpha_\beta$.

We choose to derive all our estimates in the special U^1,U^2-system employed before, but all results of the form

$w_{\alpha\beta\gamma\ldots\delta} = \mathcal{O}(Y)$ for all components of the tensor can be interpreted to stand for the equally valid and invariant statement $w_{\alpha\beta\gamma\ldots\delta}w^{\alpha\beta\gamma\ldots\delta} = \mathcal{O}(Y^2)$. Thus for the derivatives of the stresses, we have that

$$t_{\alpha\beta,\gamma_1\ldots\gamma_N} = \tau_{\alpha\beta,\gamma_1\ldots\gamma_N} + \sigma_{\alpha\beta,\gamma_1\ldots\gamma_N}u + \mathcal{O}(\varepsilon\lambda^{-N}\Omega^2)$$

where $\tau_{\alpha\beta,\gamma_1\ldots\gamma_N} = \mathcal{O}(\varepsilon\lambda^{-N})$ and $h\sigma_{\alpha\beta,\gamma_1\ldots\gamma_N} = \mathcal{O}(\varepsilon\lambda^{-N})$, and we assume these statements to stand for the analogous invariant statements.

The quantities $\ell_{\alpha\beta}$ and $e_{\alpha\beta}$ (7.2-3) will describe the deformed midsurface modulo a rigid motion. To obtain the displacement of the original undeformed midsurface Σ_0, we integrate a system of ordinary differential equations due to Gauss for $x^i{}_{,\alpha}$ and ξ^i.

(5) $x^i{}_{\|\alpha\beta} = x^i{}_{,\alpha\beta} - \left\{ {\gamma \atop \alpha\beta} \right\} x^i{}_{,\gamma} = \ell_{\alpha\beta}\xi^i; \quad \xi^i{}_{,\alpha} = -\bar{e}^{\beta\gamma}\ell_{\beta\alpha}x^i{}_{,\gamma}.$

We show that

(6) $$x^3{}_{|\alpha\beta} = \ell_{\alpha\beta} + \mathcal{O}(\Omega^2(\varepsilon+\Omega^2)/h).^7$$

By rotating and translating the deformed shell we can arrange that at P_0

$$x^i{}_{,k} = x^k{}_{,i} = \delta^k_i + \mathcal{O}(\varepsilon+\Omega^2).$$

From (1.14) we have for any point in a neighborhood of P_0

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[7] See John [2] upon which this analysis is based.

$$g_{ij} = x^k{}_{,i}x^k{}_{,j} = G_{ij} + 2\varepsilon_{ij} = \delta_{ij} + \mathcal{O}(\varepsilon+\Omega^2),$$

$$g_{ii} = x^k{}_{,i}x^k{}_{,i} = \mathcal{O}(1),$$

$$(\det x^i{}_{,k})^2 = \det g_{ii} = 1 + \mathcal{O}(\varepsilon+\Omega^2).$$

Hence, $(\det g_{ik})^{-1}$ will be bounded for small θ_0.

Consequently, $(x^s{}_{,i})^{-1}$, defined by $(x^s{}_{,i})^{-1}x^s{}_{,k} = \delta^i_k$ will be bounded. Since

$$x^s{}_{,k}x^s{}_{,ij} = [g_{ki,j} + g_{kj,i} - g_{ij,k}] = \mathcal{O}((\varepsilon+\Omega^2)/h),$$

we can now conclude that $x^s{}_{,ij} = \mathcal{O}((\varepsilon+\Omega^2)/h)$.

For any point in the shell we have

$$x^i{}_{,j}(P) = x^i{}_{,j}(P_0) + \mathcal{O}(|U^\alpha||x^i{}_{,j\alpha}| + |U^3||x^i{}_{,j3}|)$$

$$= x^i{}_{,j}(P_0) + \mathcal{O}(\Omega) = \delta^i_j + \mathcal{O}(\Omega).$$

From the identity

$$(x^i{}_{,k}-\delta^i_k) + (x^k{}_{,i}-\delta^k_i) = (g_{ik}-\delta_{ik}) - (x^j{}_{,i}-\delta^j_i)(x^j{}_{,k}-\delta^j_k) = \mathcal{O}(\Omega^2)$$

we conclude that $(x^1{}_{,1};\ x^2{}_{,2};\ x^3{}_{,3}) = 1 + \mathcal{O}(\Omega^2)$.

From the identity

$$((x^k{}_{,i})^{-1} - \delta^i_k) + (x^i{}_{,k} - \delta^i_k) = -((x^j{}_{,i})^{-1} - \delta^i_j)(x^j{}_{,k} - \delta^j_k),$$

and the fact that $(x^j{}_{,i})^{-1}(x^j{}_{,k}) = \delta^i_k$ and $\det x^j{}_{,k} = 1 + \mathcal{O}(\Omega^2)$ we have

$$(x^j{}_{,i})^{-1} = \delta^i_j + \mathcal{O}(\Omega),\ ((x^1{}_{,1})^{-1},\ (x^2{}_{,2})^{-1},\ (x^3{}_{,3})^{-1}) = 1 + \mathcal{O}(\Omega^2).$$

From (7.4) and (7.5) we have

(7) $$x^i{}_{\|\alpha\beta} = x^i{}_{|\alpha\beta} + c^\gamma_{\alpha\beta}x^i{}_{,\gamma} = \ell_{\alpha\beta}\xi^i.$$

However, $c^{\gamma}_{\alpha\beta} = \mathcal{O}(\Omega^2/\lambda)$ and on the surface $u = \pm h$ we have from (4.57)

$$\xi_3 = 1 + \mathcal{O}(\Omega^2), \quad \xi_\alpha = \mathcal{O}(\Omega), \quad \text{but}$$

$$\xi^i = G^{ij}\xi_j = \delta^j_i \xi_j + \mathcal{O}(\Omega^2), \quad \text{and consequently}$$

$$\xi^3 = 1 + \mathcal{O}(\Omega^2), \quad \xi^\alpha = \mathcal{O}(\Omega).$$

Setting $i = 3$ in (7.7) and substituting for ξ^3 we arrive at (7.6).

We introduce the quantity $\omega_{\alpha\beta}$ given by

(8) $$\omega_{\alpha\beta} = \ell_{\alpha\beta} - L_{\alpha\beta} = (x^3-f)_{|\alpha\beta} + \mathcal{O}(\Omega^2(\varepsilon+\Omega^2)/h).$$

If Ω^2 is of the same order of magnitude as ε, we shall have occasion to introduce the quantity

(9) $$\omega = x^3 - f,$$

where $\omega_{|\alpha\beta} = \omega_{\alpha\beta} + \mathcal{O}(\Omega^2(\varepsilon+\Omega^2)/h)$.

Chapter 8 - Interior Differential Equations for Shells of Variable Thickness with Surface and Body Forces

We define pseudo stresses T_i^m by

$$(1) \qquad T_i^m = (g/G)^{1/2} t_s^m g_i^s - W\delta_i^m = (W_{s_1} - W)\delta_i^m$$

$$+ (2W_{s_1} + 2W_{s_2})\varepsilon_i^m + (4W_{s_2} - 3W_{s_3})\varepsilon_s^m \varepsilon_i^s$$

$$+ 6W_{s_3} \varepsilon_j^m \varepsilon_s^j \varepsilon_i^s = t_i^m + 2(1+\nu) t_s^m t_i^s + (1-4\nu) t_j^j t_i^m$$

$$- W\delta_i^m + F(t)t^3,$$

where $W = \frac{1}{2}(1+\nu) t_s^r t_r^s - \nu t_r^r t_s^s + F(t)t^3$.

Using the stress strain relations and the equations of equilibrium we have

$$(2) \qquad T^m_{i;m} = \Big((g/G)^{1/2}\Big)_{;m} t^{ms} g_{si} + (g/G)^{1/2} t^{ms}{}_{;m} g_{si}$$

$$+ t^{ms} g_{si;m}) - W_{;i} = \sqrt{(g/G)}\Big[c^r_{mr} t^{ms} g_{si} - B^s g_{si}$$

$$- t^{rs} c^m_{rm} g_{si} - t^{mr} c^s_{rm} g_{si} + t^{ms} g_{si;m} - \tfrac{1}{2} t^{ms} g_{sm;i}\Big].$$

It follows from the definition of c^s_{rm} that

$$(3) \qquad T^m_{i;m} + (g/G)^{1/2} B^s g_{si} = 0.$$

We put down for reference,

$$(4) \quad (g/G)^{1/2} = 1 + (1-2\nu) t_i^i + F(t)t^2$$

$$g_{s\alpha} = \delta_{s\alpha}(1 + F(\eta,t)\,[\eta + t]) + 2(1+\nu) t_{s\alpha} + F(\eta,t)[t^2 + \eta t].$$

By (8.4) we have from (8.3) on the surface $u = 0$,

$$(5) \qquad T^{\beta}_{\alpha|\beta} - T^{3}_{\alpha}L^{\beta}_{\beta} + T^{3}_{\alpha,3} + \hat{B}_{\alpha} = \mathcal{O}(\Omega^2\varepsilon/\lambda),$$

where $\Omega = 2(h/\lambda)$, and $(B_k)_{u=0} = \hat{B}_k$.

From (8.1) we have on the surface $u = 0$,

$$(6a) \qquad T^{m}_{i} = t^{m}_{i} + \mathcal{O}(\varepsilon^2), \quad \text{and by (8.2)}$$

$$(6b) \qquad T^{\beta}_{\alpha|\beta} = \tau^{\beta}_{\alpha|\beta} + \mathcal{O}(\varepsilon^2/\lambda),$$

$$(6c) \qquad T^{3}_{\alpha} = t^{3}_{\alpha} + \mathcal{O}(\Omega^2\varepsilon\Omega) = \mathcal{O}(\varepsilon\Omega).$$

We have for some intermediate value v

$$(7) \; T^{3}_{\alpha} = (T^{3}_{\alpha})_{u=0} + u(T^{3}_{\alpha,3})_{u=0} + \tfrac{1}{2}u^2(T^{3}_{\alpha,33})_{u=0} + \tfrac{1}{6}u^3(T^{3}_{\alpha,333})_{u=v}.$$

Evaluating this expression for $u = +h$ and $u = -h$ we have

$$\frac{1}{2h}(T^{3+}_{\alpha} - T^{3-}_{\alpha}) = (T^{3}_{\alpha,3})_{u=0} + \tfrac{1}{6}h^2(T^{3}_{\alpha,333})_{u=v'} \quad \text{or}$$

$$(T^{3}_{\alpha,3})_{u=0} = \frac{1}{2h}(T^{3+}_{\alpha} - T^{3-}_{\alpha}) + \mathcal{O}(\Omega^2\varepsilon/\lambda).$$

We obtain from (8.5)

$$\tau^{\beta}_{\alpha|\beta} + \frac{1}{2h}(t^{3+}_{\alpha} - t^{3-}_{\alpha}) + \hat{B}_{\alpha} = \mathcal{O}(\Omega^2\varepsilon/\lambda).$$

Writing the third equilibrium equation in terms of the pseudo stresses we have for $u = 0$:

$$(8) \qquad \hat{B}_3 + (T^{i}_{3;i})_{u=0} = T^{\alpha}_{3|\alpha} - T^{3}_{3}L^{\alpha}_{\alpha} + T^{\alpha}_{\beta}L^{\beta}_{\alpha} + T^{3}_{3,3} + \hat{B}_3$$

$$= \mathcal{O}(\Omega^2\varepsilon\Omega/\lambda).$$

Here by (8.6c) and (6.1)

$$T^{\alpha}_{3|\alpha} = t^{\alpha}_{3|\alpha} + \mathcal{O}(\Omega^2\varepsilon\Omega/\lambda), \quad T^{\alpha}_{\beta} = t^{\alpha}_{\beta} + \mathcal{O}(\Omega^2\varepsilon),$$

$$t^{\alpha\pm}_{3} = E^{\alpha\beta}T^{3\pm}_{\beta} + \mathcal{O}(\Omega^2\varepsilon\Omega).$$

Consequently, we have using a Taylor expansion

(8a)
$$(T^{\alpha}_{3})_{u=0} - \frac{T^{\alpha+}_{3}+T^{\alpha-}_{3}}{2} = E^{\alpha\beta}(T^{3}_{\beta})_{u=0} - E^{\alpha\beta}\frac{(T^{3+}_{\beta}+T^{3-}_{\beta})}{2} + \mathcal{O}(\Omega^2\varepsilon\Omega)$$

$$= -\tfrac{1}{2}h^2E^{\alpha\beta}(T^{3}_{\beta,33})_{u=0} + \tfrac{1}{24}h^4E^{\alpha\beta}(T^{3}_{\beta,3333})_{u=v} + \mathcal{O}(\Omega^2\varepsilon\Omega)$$

$$= -\tfrac{1}{2}h^2E^{\alpha\beta}(T^{3}_{\beta;33})_{u=0} + \mathcal{O}(\Omega^2\varepsilon\Omega)$$

$$= \tfrac{1}{2}h^2E^{\alpha\beta}(T^{\gamma}_{\beta;\gamma3} + B_{\beta;3})_{u=0} + \mathcal{O}(\Omega^2\varepsilon\Omega)$$

$$= \tfrac{1}{2}h^2E^{\alpha\beta}(t^{\gamma}_{\beta;\gamma3} + B_{\beta;3})_{u=0} + \mathcal{O}(\Omega^2\varepsilon\Omega)$$

$$= \tfrac{1}{2}h^2E^{\alpha\beta}(E^{\gamma\mu}t_{\mu\beta;\gamma3} + B_{\beta;3})_{u=0} + \mathcal{O}(\Omega^2\varepsilon\Omega)$$

$$= \tfrac{1}{2}h^2(E^{\alpha\beta}E^{\gamma\mu}\sigma_{\mu\beta|\gamma} + B^{\alpha}{}_{;3})_{u=0} + \mathcal{O}(\Omega^2\varepsilon\Omega)$$

$$= \tfrac{1}{2}h^2(\sigma^{\alpha\gamma}{}_{|\gamma} + (B^{\alpha}{}_{,3})_{u=0}) + \mathcal{O}(\Omega^2\varepsilon\Omega).$$

Now performing covariant differentiation with respect to the metric in the undeformed surface gives

$$(T^{\alpha}_{3|\alpha})_{u=0} = (\tfrac{1}{2}h^2\sigma^{\alpha\gamma}{}_{|\gamma})_{|\alpha} + \frac{(t^{3\alpha+}+t^{3\alpha-})_{|\alpha}}{2} + \tfrac{1}{2}\left[h^2(\hat{B}^{\alpha}{}_{,3})\right]_{|\alpha}$$

$$+ \mathcal{O}(\Omega^2\varepsilon\Omega/\lambda), \quad \text{where} \quad (B^{k}{}_{,i_1\ldots i_N})_{u=0} = \hat{B}^{k}{}_{,i_1\ldots i_N}.$$

Expanding the quantity (T^3_3+W) along a fiber perpendicular to Σ_0, we have

(9)
$$\frac{1}{2h}\left[(T^{3+}_{3} + W^{+}) - (T^{3-}_{3} + W)\right] = (T^{3}_{3,3} + W_{,3})_{u=0}$$

$$+ \tfrac{1}{6}h^2(T^{3}_{3,333} + W_{,333})_{u=0} + \mathcal{O}(\Omega^2\varepsilon\Omega^2/h).$$

Calculating $T^3_3 + W$ in terms of the stresses t^i_j we have from (8.1)

$$T^3_3 + W = t^3_3 + 2(1+\nu)t^3_s t^s_3 + (1-4\nu)t^j_j t^3_3 + F(t)t^3,$$

using an identity similar to (4.42) we have by (6.1)

$$\left[(T^{3+}_3 + W^+) - (T^{3-}_3 + W^-)\right]$$
$$= (t^{3+}_3 - t^{3-}_3) + (1-4\nu)(t^{\alpha+}_\alpha - t^{\alpha-}_\alpha)\left[\tfrac{1}{2}(t^{3+}_3 + t^{3-}_3)\right] + \mathcal{O}(\Omega^2 \varepsilon \Omega^2).$$

Since

$$(t^{\alpha+}_\alpha - t^{\alpha-}_\alpha) = 2h(t^\alpha_{\alpha,3})_{u=0} + \mathcal{O}(\Omega^2 \varepsilon),$$

we have

$$(10) \qquad \frac{1}{2h}\left[(T^{3+}_3 + W^+) - (T^{3-}_3 + W^-)\right] = \frac{1}{2h}(t^{3+}_3 - t^{3-}_3)$$
$$+ (1-4\nu)(t^\alpha_{\alpha,3})_{u=0}\left[\tfrac{1}{2}(t^{3+}_3 + t^{3-}_3)\right] + \mathcal{O}(\Omega^2 \varepsilon \Omega^2/h).$$

We now seek an expansion for $(W_{,3})_{u=0}$ in terms of the stresses and the boundary conditions. Dividing by $(g/G)^{1/2}$ and differentiating (1.6) with respect to U^3, we have

$$(11) \qquad (W_{,3})_{u=0} = ((g/G)^{1/2} t^{mk} \varepsilon_{mk,3})_{u=0}$$
$$= (t^{mk}\varepsilon_{mk,3})_{u=0} + \mathcal{O}(\Omega^2 \varepsilon \Omega^2/h)$$
$$= (t^{\alpha\beta}\varepsilon_{\alpha\beta,3})_{u=0} - \nu(t^{33} t^\alpha_{\alpha,3})_{u=0} + \mathcal{O}(\Omega^2 \varepsilon \Omega^2/h).$$

From the expression for $\ell_{\alpha\beta}$ (7.3), we find using (8.4), (1.14), and the estimates for the stresses and their derivatives from (6.1),

$$\omega_{\alpha\beta} = \ell_{\alpha\beta} - L_{\alpha\beta} = -\tfrac{1}{2}(g_{\alpha\beta;3})_{u=0} + \mathcal{O}(\Omega^2\varepsilon/h)$$

$$= -(\varepsilon_{\alpha\beta;3})_{u=0} + \mathcal{O}(\Omega^2\varepsilon/h) = -(\varepsilon_{\alpha\beta,3})_{u=0} + \mathcal{O}(\Omega^2\varepsilon/h).$$

On the other hand, from (1.14, 6.1, 7.1, 7.2) we find

$$(\varepsilon_{\alpha\beta,3})_{u=0} = (1+\nu)\sigma_{\alpha\beta} - \nu\sigma^{\delta}_{\delta}E_{\alpha\beta} + \mathcal{O}(\Omega^2\varepsilon/h).$$

Therefore, from (1.4) we have

(12a) $$\omega_{\alpha\beta} = -(1+\nu)\sigma_{\alpha\beta} + \nu\sigma^{\gamma}_{\gamma}E_{\alpha\beta} + \mathcal{O}(\Omega^2\varepsilon/h),$$

(12b) $$\omega_{\alpha\alpha} = -(1-\nu)\sigma_{\alpha\alpha} + \mathcal{O}(\Omega^2\varepsilon/h).$$

From (8.6a)

$$(T^3_3)_{u=0} = (t^3_3)_{u=0} + \mathcal{O}(\Omega^2\varepsilon)$$

$$= \frac{[t^{3+}_3 + t^{3-}_3]}{2} + \mathcal{O}(\Omega^2\varepsilon).$$

Upon substituting into (8.11), we have

(13) $$(W_{,3})_{u=0} = -\tau^{\alpha\beta}\omega_{\beta\alpha} + \frac{\nu}{1-\nu}\omega^{\alpha}_{\alpha}\left[\tfrac{1}{2}(t^{33+}+t^{33-})\right] + \mathcal{O}(\Omega^2\varepsilon\Omega^2/h).$$

Substituting for $(t^{\alpha}_{\alpha,3})_{u=0} = \sigma^{\alpha}_{\alpha}$ in (8.10) we have from (8.12b)

(14) $$\frac{1}{2h}\left[(T^{3+}_3 + W^+) - (T^{3-}_3 + W^-)\right] = \frac{1}{2h}(t^{3+}_3 - t^{3-}_3)$$

$$- \frac{(1-4\nu)}{1-\nu}\omega^{\alpha}_{\alpha}\left[\tfrac{1}{2}(t^{3+}_3 + t^{3-}_3)\right] + \mathcal{O}(\Omega^2\varepsilon\Omega^2/h).$$

Next we note that for $u = 0$:

$$(15) \qquad T^{3}_{3,333} + W_{,333} = T^{3}_{3;333} + \mathcal{O}(\Omega^2\varepsilon/h\lambda^2)$$

$$= -E^{\beta\alpha}T_{3\alpha;\beta 33} - \hat{B}_{3,33} + \mathcal{O}(\Omega^2\varepsilon/h\lambda^2)$$

$$= E^{\beta\alpha}T^{\gamma}_{\alpha;\beta\gamma 3} + E^{\beta\alpha}\hat{B}_{\alpha,\beta 3} - \hat{B}_{3,33} + \mathcal{O}(\Omega^2\varepsilon/h\lambda^2)$$

$$= \sigma^{\beta\gamma}{}_{|\beta\gamma} + \hat{B}_{\alpha,\alpha 3} - \hat{B}_{3,33} + \mathcal{O}(\Omega^2\varepsilon/h\lambda^2).$$

Substituting (8.15), (8.14), (8.13) into (8.9) and substituting for $T^{3}_{3,3}$ in (8.8) we conclude that

$$(16) \quad \tfrac{1}{2}(h^2\sigma^{\alpha\gamma}{}_{|\gamma})_{|\alpha} - \frac{h^2}{6}\sigma^{\beta\gamma}{}_{|\beta\gamma} + \tfrac{1}{2}(h^2\hat{B}^{\alpha}{}_{,3})_{|\alpha} - \tfrac{1}{6}h^2(\hat{B}_{\alpha,\alpha 3}-\hat{B}_{3,33})$$

$$+ \hat{B}_3 + \tfrac{1}{2}(t^{3\alpha+}+t^{3\alpha})_{|\alpha} - \tfrac{1}{2}(t^{33+}+t^{33-})(L^{\alpha}_{\alpha}+\omega^{\alpha}_{\alpha}) + \tau^{\alpha}_{\beta}(L^{\beta}_{\alpha}+\omega^{\beta}_{\alpha})$$

$$+ \frac{1}{2h}(t^{3+}_{3} - t^{3-}_{3}) + \frac{1-2\nu}{1-\nu}(t^{3+}_{3} + t^{3-}_{3})\omega^{\alpha}_{\alpha} = \mathcal{O}(\Omega^2\varepsilon h/\lambda^2).$$

The Codazzi equations for the deformed midsurface and undeformed midsurface of the shell are respectively:

$$\ell_{\alpha\beta\|\gamma} = \ell_{\alpha\gamma\|\beta}, \quad L_{\alpha\beta|\gamma} = L_{\alpha\gamma|\beta}.$$

We have in terms of $\omega_{\alpha\beta}$ and $c^{\mu}_{\alpha\beta}$:

$$0 = \omega_{\alpha\beta|\gamma} - \omega_{\alpha\gamma|\beta} - (L_{\mu\beta} + \omega_{\mu\beta})c^{\mu}_{\alpha\gamma} + (L_{\mu\gamma} + \omega_{\mu\gamma})c^{\mu}_{\alpha\beta}.$$

This gives us two non-trivial equations:

$$(17) \qquad \omega_{\alpha\beta|\gamma} - \omega_{\alpha\gamma|\beta} = \mathcal{O}(\Omega^2\varepsilon/h\lambda).$$

We write the Gauss equations for the curvature as

$$
\begin{aligned}
\ell_{11}\ell_{22} - \ell_{12}\ell_{12} &= \tfrac{1}{2}(e_{12,21} + e_{21,12} - e_{11,22} - e_{22,11}) \\
&\quad + e_{\lambda\mu}\{\overline{\{{\mu \atop 12}\}}\overline{\{{\lambda \atop 21}\}} - \overline{\{{\mu \atop 11}\}}\overline{\{{\lambda \atop 22}\}}\} \\
&= \tfrac{1}{2}(e_{12|21} + e_{21|12} - e_{11|22} - e_{22|11}) \\
&\quad + \tfrac{1}{2}e^{\alpha}_{\alpha}(L_{11}L_{22} - L_{12}L_{21}) + e_{\lambda\mu}(c^{\lambda}_{12}c^{\mu}_{21} - c^{\lambda}_{11}c^{\mu}_{22}),
\end{aligned}
$$

and express the quantities $e_{\alpha\beta}$ in terms of $\tau_{\alpha\beta}$ and $\sigma_{\alpha\beta}$. For this purpose we employ a set of approximate "constituent" equations. From (1.14) using (6.1, 7.1, 7.2) we have

$$
\tfrac{1}{2}(e_{\alpha\beta} - E_{\alpha\beta}) = (1+\nu)\tau_{\alpha\beta} - \nu(\tau^{\gamma}_{\gamma} + \tfrac{1}{2}(t^{3+}_{3} + t^{3-}_{3}))E_{\alpha\beta} + \mathcal{O}(\Omega^2\varepsilon).
$$

Combining the constituent equations and the Gauss equations, we conclude using (8.7) that

$$
\begin{aligned}
L_{11}\omega_{22} + L_{22}\omega_{11} - L_{12}\omega_{21} - L_{21}\omega_{12} &= -(1+\nu)\tau_{\lambda\lambda|\mu\mu} \\
&\quad + \nu\tau_{\lambda\lambda|\mu\mu} + \nu\tfrac{1}{2}(t^{3+}_{3} + t^{3-}_{3})_{|\mu\mu} - (1+\nu)\Big[\hat{B}_{\mu} + \\
&\quad + \frac{t^{3+}_{\mu} - t^{3-}_{\mu}}{2h}\Big]_{|\mu} - \tfrac{1}{2}(\omega_{11}\omega_{22} - \omega_{12}\omega_{21}) + \mathcal{O}(\Omega^2\varepsilon/\lambda^2).
\end{aligned}
$$

Writing the equation invariantly, we have

$$
\begin{aligned}
(18)\quad & \tau^{\lambda}_{\lambda}|^{\mu}_{\mu} + L^{\alpha}_{\alpha}\omega^{\beta}_{\beta} - L^{\beta}_{\alpha}\omega^{\alpha}_{\beta} + \tfrac{1}{2}(\omega^{\alpha}_{\alpha}\omega^{\beta}_{\beta} - \omega^{\alpha}_{\beta}\omega^{\beta}_{\alpha}) - \tfrac{\nu}{2}(t^{3+}_{3} + t^{3}_{3})|^{\mu}_{\mu} \\
& + (1+\nu)\Big[\hat{B}_{\alpha} + \Big(\frac{t^{3+}_{\alpha} - t^{3-}_{\alpha}}{2h}\Big)\Big]\Big|^{\alpha} = \mathcal{O}(\Omega^2\varepsilon/\lambda^2).
\end{aligned}
$$

It remains to express the terms $t^{3\pm}_{k}$ in terms of the applied surface forces, and the U^3 derivative of the stresses $(t^{\alpha}_{\beta,3})_{u=0} = \sigma^{\alpha}_{\beta}$ in terms of the curvature changes ω^{α}_{β}.

From equations (8.12) and (8.17) we have

$$(19)\qquad \sigma_{\alpha\beta} = -\frac{1}{1+\nu}\omega_{\alpha\beta} - \frac{\nu}{1-\nu^2}\omega^{\gamma}_{\gamma}E_{\alpha\beta} + \mathcal{O}(\Omega^2\varepsilon/h),$$

$$(20)\qquad \sigma^{\alpha\gamma}\big|_{\gamma} = -\frac{1}{1+\nu}\omega^{\alpha\gamma}\big|_{\gamma} - \frac{\nu}{1-\nu^2}\omega^{\mu}_{\mu|\gamma}E^{\alpha\gamma} + \mathcal{O}(\Omega^2\varepsilon/h\lambda)$$

$$= -\frac{\nu}{1-\nu^2}\omega^{\gamma|\alpha}_{\gamma} + \mathcal{O}(\Omega^2\varepsilon/h\lambda).$$

From equation (4.57) we have for the surface forces

$$(21)\qquad s^{\pm}_{\beta} = t^{3\pm}_{\beta} \mp h_{,\alpha}t^{\alpha\pm}_{\beta} + \mathcal{O}(\Omega^2\varepsilon\Omega/\lambda),$$

$$s^{+}_{3} - s^{-}_{3} = (t^{3+}_{3} - t^{3-}_{3}) - h_{,\alpha}(t^{\alpha+}_{3} + t^{\alpha-}_{3}) + \mathcal{O}(\Omega^2\varepsilon\Omega^2),$$

$$s^{+}_{3} + s^{-}_{3} = (t^{3+}_{3} + t^{3-}_{3}) + \mathcal{O}(\Omega^2\varepsilon).$$

Consequently,

$$(22a)\qquad \frac{(t^{3+}_{\alpha} - t^{3-}_{\alpha})}{2h} = \frac{(t^{\beta+}_{\alpha} + t^{\beta-}_{\alpha})}{2}\,\frac{h_{,\beta}}{h} + \frac{(s^{+}_{\alpha} - s^{-}_{\alpha})}{2h} + \mathcal{O}(\Omega^2\varepsilon\Omega/h)$$

$$= \tau^{\beta}_{\alpha}\frac{h_{,\beta}}{h} + \frac{(s^{+}_{\alpha} - s^{-}_{\alpha})}{2h} + \mathcal{O}(\Omega^2\varepsilon\Omega/h).$$

Similarly, we have

$$(22b)\qquad \tfrac{1}{2}(t^{3+}_{3} + t^{3-}_{3}) = \tfrac{1}{2}(s^{+}_{3} + s^{-}_{3}) + \mathcal{O}(\Omega^2\varepsilon),$$

$$(22c)\qquad \tfrac{1}{2h}(t^{\alpha+}_{\beta} - t^{\alpha-}_{\beta}) = \sigma^{\alpha}_{\beta} + \mathcal{O}(\Omega^2\varepsilon/h),$$

$$(22d)\qquad \tfrac{1}{2h}(t^{3+}_{3} - t^{3-}_{3}) = \tfrac{1}{2h}(s^{+}_{3} - s^{-}_{3}) + \sigma^{\alpha\beta}h_{,\alpha}h_{,\beta} + \mathcal{O}(\Omega^2\varepsilon\Omega^2/h),$$

$$(22e)\qquad \tfrac{1}{2}(t^{3+}_{\alpha} + t^{3-}_{\alpha}) = hh_{,\beta}\sigma^{\beta}_{\alpha} + \tfrac{1}{2}(s^{+}_{\alpha} + s^{-}_{\alpha}) + \mathcal{O}(\Omega^2\varepsilon\Omega).$$

Inserting equations (8.19-22) in equation (8.16) we have, upon rearranging the terms, reintroducing E, Young's modulus, and expressing the body forces as integrals

$$(23a)\quad -\frac{E}{1-\nu^2}\frac{1}{3h}\left\{(h^3\omega^\alpha_\alpha|^\beta_\beta - (1-\nu)\left[(h^3)|^\alpha_\alpha\omega^\beta_\beta - (h^3)|^\alpha_\beta\omega^\beta_\alpha\right]\right\}$$

$$+ \tau^\alpha_\beta(L^\beta_\alpha + \omega^\beta_\alpha) - \tfrac{1}{2}(S^+_3 + S^-_3)(L^\alpha_\alpha - (\tfrac{1-3\nu}{1-\nu})\omega^\alpha_\alpha)$$

$$+ \tfrac{1}{2}(S^+_\alpha + S^-_\alpha)|^\alpha + \frac{1}{2h}(S^+_3 - S^-_3) + \frac{1}{2h}\int_{-h}^{h} B_3 du + \frac{1}{2h}(\int_{-h}^{h} uB_\alpha du)|^\alpha$$

$$= \mathcal{O}(E\Omega^2\varepsilon h/\lambda^2).$$

From equations (8.7) and (8.18) we have respectively

$$(23b)\quad (h\tau^\beta_\alpha)|_\beta + h\hat{B}_\alpha + \tfrac{1}{2}(S^+_\alpha - S^-_\alpha) = \mathcal{O}(E\Omega^2\varepsilon h/\lambda),\quad \alpha = 1,2;$$

$$(23c)\quad \frac{1}{E}\left\{\tau^\lambda_\lambda|^\mu_\mu - \tau^\lambda_\mu|^\mu_\lambda(1+\nu)\right\} + L^\alpha_\alpha\omega^\beta_\beta - L^\beta_\alpha\omega^\alpha_\beta$$

$$+ \tfrac{1}{2}(\omega^\alpha_\alpha\omega^\beta_\beta - \omega^\beta_\alpha\omega^\alpha_\beta) - \nu\frac{(S^+_3 + S^-_3)|^\mu_\mu}{2E} = \mathcal{O}(\Omega^2\varepsilon/\lambda^2).$$

We complete the set of six approximate equations for the unknowns ω^β_α and τ^α_β with the equation

$$(23d)\qquad \omega^\beta_\alpha|_\gamma - \omega^\beta_{\gamma|\alpha} = \mathcal{O}(\Omega^2\varepsilon/h\lambda),$$

which provides two non-trivial equations.

As a consequence of (8.8a) and (8.20) we have

$$(t^3_\beta)_{u=0} = E_{\beta\alpha}(t^\alpha_3)_{u=0}$$

$$= -\frac{h^2}{2(1-\nu^2)}\omega^\gamma_\beta\Big|_\gamma + \frac{t^{3+}_\beta + t^{3-}_\beta}{2} + \frac{h^2}{2}\hat{B}_{\beta,3} + \mathcal{O}(E\Omega^2\varepsilon h/\lambda).$$

Since $t^3_\beta = \mathcal{O}(E\varepsilon\Omega)$ and $t^3_{\beta,333} = \mathcal{O}(E\varepsilon/\lambda^3)$ we find for t^3_β the approximation

$$t^3_\beta = -\frac{E(h^2-u^2)}{2(1-\nu^2)}\omega^\gamma_\beta\Big|_\gamma + u\frac{(\tau^\alpha_\beta h_{,\alpha})}{h} - hE\left\{\frac{h_{,\alpha}}{1+\nu}\omega^\alpha_\beta + \frac{\nu h_{,\beta}}{1-\nu^2}\omega^\alpha_\alpha\right\}$$

$$+ u\frac{(S^+_\beta - S^-_\beta)}{2h} + \frac{(S^+_\beta + S^-_\beta)}{2} + \frac{(h^2-u^2)}{2}\hat{B}_{\beta,3} + \mathcal{O}(E\Omega^2\varepsilon h/\lambda).$$

For t^3_3 we have,

$$t^3_3 = \tfrac{1}{2}(S^+_3 + S^-_3) + \mathcal{O}(E\Omega^2\varepsilon).$$

Chapter 9 - Interior Differential Equations for Shallow Shells of Variable Thickness with Restricted Body and Surface Forces

We shall assume that there exists a potential function $S = S(U^1, U^2)$, such that the resultant of the surface tractions on the midsurface is given by

$$\tfrac{1}{2}(S^+_\alpha - S^-_\alpha) = S_{,\alpha} = \mathcal{O}(E\varepsilon\Omega),$$

where,

$$S_{,\alpha_1\ldots\alpha_p} = \mathcal{O}\,(\varepsilon\Omega/\lambda^{p-1}), \quad p = 0,\ldots,N+1.$$

For the other surface forces we have from (3.5)

$$(S^{3+} - S^{3-})_{,\alpha_1\ldots\alpha_p} = \mathcal{O}\,(\varepsilon\Omega^2/\lambda^p), \quad p = 0,\ldots,N$$

$$(S^{3+} + S^{3-})_{,\alpha_1\ldots\alpha_p} = \mathcal{O}\,(\varepsilon/\lambda^p), \quad p = 0,\ldots,N,$$

and for the body forces perpendicular to the midsurface Σ_0 we have from (3.5), $B_3 = \mathcal{O}(\varepsilon\Omega/\lambda)$ and

$$B_{3,i_1\ldots i_p} = \begin{cases} \mathcal{O}(\varepsilon/\lambda^{p+1}) & \text{if an odd number of the } i_1,\ldots,i_p \text{ equals } 3, \\ \mathcal{O}(\varepsilon/\lambda^p h) & \text{otherwise.} \end{cases}$$

We assume that the body forces in the plane of the shell are negligible

$$B_{\alpha,i_1\ldots i_p} = \mathcal{O}(\varepsilon\Omega^4/\lambda^{p+1}), \quad p = 0,\ldots,N.$$

From (8.23) we have under the assumptions of this chapter, for $\Omega = 2(h/\lambda)$

$$\text{(1a)}\quad (h\tau^{\beta}_{\alpha})|_{\beta} + S_{,\alpha} = \mathcal{O}(E\Omega^2\varepsilon h/\lambda),$$

$$\text{(1b)}\quad \omega^{\beta}_{\alpha|\gamma} - \omega^{\beta}_{\gamma|\alpha} = \mathcal{O}(\Omega^2\varepsilon/h\lambda),$$

$$\text{(1c)}\quad \frac{1}{E}\left[\tau^{\lambda}_{\lambda}|^{\mu}_{u} - \tau^{\lambda}_{\mu}|^{\mu}_{\lambda}(1+\nu)\right] + L^{\alpha}_{\alpha}\omega^{\beta}_{\beta} - L^{\beta}_{\alpha}\omega^{\alpha}_{\beta}$$
$$+ \frac{1}{2}(\omega^{\alpha}_{\alpha}\omega^{\beta}_{\beta} - \omega^{\beta}_{\alpha}\omega^{\alpha}_{\beta}) - \nu\frac{(S^{+}_{3} + S^{-}_{3})|^{\mu}_{u}}{2E} = \mathcal{O}(\Omega^2\varepsilon/\lambda^2),$$

$$\text{(1d)}\quad \frac{-E}{1-\nu^2}\frac{1}{3h}\left[\left(h^3\omega^{\alpha}_{\alpha}\right)\Big|^{\beta}_{\beta} - (1-\nu)\left((h^3)|^{\alpha}_{\alpha}\omega^{\beta}_{\beta} - (h^3)|^{\alpha}_{\beta}\omega^{\beta}_{\alpha}\right)\right] + \frac{1}{2}(S^{+}_{\alpha}+S^{-}_{\alpha})|^{\alpha}$$
$$+ \tau^{\alpha}_{\beta}(L^{\beta}_{\alpha} + \omega^{\beta}_{\alpha}) - \frac{1}{2}(S^{+}_{3} + S^{-}_{3})(L^{\alpha}_{\alpha} - (\frac{1-3\nu}{1-\nu})\omega^{\alpha}_{\alpha})$$
$$+ \frac{1}{2h}(S^{+}_{3} - S^{-}_{3}) + \frac{1}{2h}\int_{-h}^{h} B_3\,du = \mathcal{O}(E\Omega^2\varepsilon h/\ {}^2).$$

If we assume that H/R is close to ε in magnitude, we can derive equations for shallow shells which resemble those of v. Karman and Föppl[8] derived for plates of constant thickness.

If we assume an Airy stress function to exist, then it will satisfy equation (8.4a) to the required order of error.[9] For Φ the Airy stress function we set

$$h\tau^{\alpha}_{\beta} = (\delta^{\alpha}_{\beta}\delta^{\gamma}_{\delta} - \delta^{\alpha}_{\delta}\delta^{\gamma}_{\beta})\Phi|^{\delta}_{\gamma} - SE^{\alpha}_{\beta}.$$

Performing the covariant surface differentiation, we have

$$(h\tau^{\beta}_{1})|_{\beta} + S_{,1} = \Phi|^{2}_{21} - \Phi|^{2}_{12} = \Phi|^{\alpha}R_{\alpha 212}$$
$$(h\tau^{\beta}_{2})|_{\beta} + S_{,2} = \Phi|^{1}_{12} - \Phi|^{1}_{21} = \Phi|^{\alpha}R_{\alpha 121},$$

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[8] See v. Karman [4].

[9] See Green [1].

where $R_{\alpha\beta\delta\gamma}$ is the Riemann tensor for the surface Σ_0.

We show that

$$\Phi|^{\alpha}R_{\alpha\beta\delta\gamma} = \mathcal{O}(E\Omega^2(\varepsilon\Omega+\Omega^3)).$$

At P_0, we have $f(P_0) = f_{,\alpha}(P_0) = 0$, and we set $\Phi(P_0) = \Phi_{,\alpha}(P_0) = 0$. By the assumptions made on f (2.3), we have $R_{\alpha\beta\delta\gamma} = \mathcal{O}(\Omega^2/\lambda^2)$. Hence, it will be sufficient to show that

$$(2) \qquad \Phi|_{\alpha} = \Phi_{,\alpha} = \mathcal{O}(E\varepsilon\Omega\lambda^2 + E\Omega^3\lambda^2).$$

Expanding $\Phi_{,\alpha}$ about the point P_0, we have for some intermediate value P'

$$(3) \qquad \Phi_{,\alpha} = \Phi|_{\alpha} = \Phi_{,\alpha}(P_0) + \mathcal{O}(|U^{\beta}||\phi_{,\alpha\beta}|_{P=P'}),$$

and for $\Phi_{,\alpha\beta}$ we have

$$(4) \qquad \Phi_{,\alpha\beta} = \Phi|_{\alpha\beta} + \{{}^{\gamma}_{\alpha\beta}\}\Phi_{,\gamma}.$$

Since $\Phi|_{\alpha\beta}(P) = \mathcal{O}(E\varepsilon h)$, we have $\Phi_{,\alpha\beta}(P_0) = \mathcal{O}(E\varepsilon h)$. The expansion

$$\Phi_{,\alpha} = \Phi_{,\alpha}(P_0) + U^{\beta}\Phi_{,\alpha\beta}(P_0) + \mathcal{O}(|U^{\delta}||U^{\beta}||\Phi_{,\alpha\beta\delta}|),$$

together with the fact that $\Phi_{,\alpha\beta\delta}$ is bounded, gives

$$\Phi_{,\alpha} = \mathcal{O}(E\varepsilon\Omega + E\lambda^2).$$

Substituting this result into (9.4) gives, using (3.7b) to estimate $\{{}^{\gamma}_{\alpha\beta}\}$,

$$\Phi_{,\alpha\beta} = \mathcal{O}(E\varepsilon h + E\lambda\Omega^2),$$

and consequently from (9.3)

$$\Phi_{,\alpha} = \mathcal{O}(E\varepsilon h\lambda + E\lambda^2\Omega^2).$$

Substituting into (9.4) gives

$$\Phi_{,\alpha\beta} = \mathcal{O}(E\varepsilon h + E\Omega^3\lambda),$$

inserting this into (9.3) for $\Phi_{,\alpha\beta}$ gives (9.2).

Recalling from (7.9) that $\omega|_{\alpha\beta} = \omega_{\alpha\beta} + \mathcal{O}(\Omega^4/h) = \ell_{\alpha\beta} - L_{\alpha\beta} = x^3|_{\alpha\beta} - f|_{\alpha\beta} + \mathcal{O}((\varepsilon+\Omega^2)\Omega^2/h)$, we can reduce the equations (9.1) to two equations for two unknown functions Φ and ω.[10]

For notational covvenience we define the invariant differential operators $\Diamond^4$ and Δ

$$\Diamond^4(u,v) = u|^{\alpha}_{\alpha}v|^{\beta}_{\beta} - u|^{\alpha}_{\beta}v|^{\beta}_{\alpha} = \Diamond^4(v,u)$$

$$\Delta u = u|^{\alpha}_{\alpha}.$$

Under the assumption that the shell is shallow, that the surface traction forces can be derived from a potential function, and the body forces in the plane of the shell are negligible, we have from (9.1)

$$\text{(5a)}\quad \frac{1}{E}[\Delta(h^{-1}\Delta\Phi) - (1+\nu)\Diamond^4(h^{-1},\Phi)] + \Diamond^4(f,\omega) + \frac{1}{2}\Diamond^4(\omega,\omega)$$

$$- \frac{\Delta}{E}(\frac{\nu}{2}(S_3^+ + S_3^-) + (1-\nu)\frac{S}{h}) = \mathcal{O}(\Omega^2(\varepsilon+\Omega^2)/\lambda^2),$$

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[10] For an engineer's approach to this problem, together with some series solution to the resultant equations, see Mansfield [5].

$$(5b)\quad -\frac{E}{1-\nu^2}\lceil\tfrac{1}{3}[\Delta(h^3\Delta\omega) - (1-\nu)\Diamond^4(h^3,\omega)] + \Diamond^4(\Phi,\omega+f) + h\frac{(S^+_\alpha+S^-_\alpha)|^\alpha}{2}$$

$$- S\Delta(f+\omega) - \frac{h(S^+_3 + S^-_3)}{2}\Delta(f - \frac{1-3\nu}{1-\nu}\omega) + \tfrac{1}{2}(S^+_3 - S^-_3)$$

$$+ \tfrac{1}{2}\int_{-h}^{h} B_3 du = \mathcal{O}\,(E\Omega^2(\varepsilon\Omega^2 + \Omega^4)),$$

where $\Omega = 2h/\lambda$, E = Young's modulus, ν = Poisson's ratio. $\omega = x^3-f$, x^3 = coordinate of the deformed midsurface, f = the X^3 coordinate of the undeformed midsurface, h = the variation in thickness of the undeformed shell, Φ = the Airy stress function, $S^\pm_k$ = the surface loads in the U^α coordinate system per unit deformed surface area on the deformed shell, S = a potential function such that $S_{,\alpha} = \frac{1}{2}(S^+_\alpha - S^-_\alpha)$, and B_3 = body force per unit deformed volume in the U^k coordinate system.

Note that by our assumptions on the magnitude of the applied forces, the forces normal to the midsurface can be of order ε. The resultant normal force, however, must be considerably smaller. In other words we have

$$S^+_3,\ S^-_3 = \mathcal{O}\,(\varepsilon),$$

$$S^+_3 - S^-_3 = \mathcal{O}\,(\varepsilon\Omega^2).$$

General Inequalities for Hilbert Norms of Functions on a Slab

These results are independent of the theory of elasticity. Greek subscripts $\alpha, \beta, \ldots$ range over the values $1, 2$. Repeated subscripts or superscripts are to be summed.

Let λ and H be fixed positive quantities. The function $w = w(U^1, U^2, U^3)$ is defined and of class C^2 in the region $M: U^\alpha U^\alpha < \lambda^2, \quad |U^3| < H$.

We define the symbols "$\mathcal{O}$" and "o" as follows:

$A = \mathcal{O}(B)$ with $B \geq 0$ means that there exists a universal constant K such that $|A| \leq KB$.

$A = \mathcal{O}(B) + o(C)$ with $B \geq 0$ and $C \geq 0$ means there exists a universal function $K(\delta)$ such that for each positive δ the inequality

$$|A| \leq K(\delta)B + \delta C$$

is satisfied.

We define:

$$\|w\|^2 = \iiint_M |w|^2 dU^1 dU^2 dU^3 = \int_M |w|^2 dU,$$

$$\overline{w} = \sup_M |w|,$$

$$w_{,\alpha} = \frac{\partial w}{\partial U^\alpha},$$

$$\Delta w = w_{,11} + w_{,22}.$$

We employ an auxiliary function

$$\phi(U^1,U^2) = (1 - \lambda^{-2}U^\alpha U^\alpha)^2.$$

Clearly for $U = (U^1,U^2,U^3)$ in M

(A0) $\quad 0 \leq \phi \leq 1, \quad \sqrt{\phi_{,\alpha}\phi_{,\alpha}} \leq 2\lambda^{-1}\sqrt{\phi}, \quad \sqrt{\phi_{,\alpha\beta}\phi_{,\alpha\beta}} \leq 8\lambda^{-2}.$

ϕ and its first derivatives vanish for $U^\alpha U^\alpha = \lambda^2$, enabling us to carry out integration by parts with respect to U^α without producing boundary terms.

Proofs of the following inequalities are based on integration by parts.

LEMMA

(A1) $$\|\phi_{,\alpha}w_{,\beta}\| = \mathcal{O}(\lambda^{-2}\|w\|) + \sigma(\|\phi\Delta w\|).$$

PROOF

$$\begin{aligned}\sum_{\alpha,\beta}\|\phi_{,\alpha}w_{,\beta}\|^2 &= \int_M \phi_{,\alpha}\phi_{,\alpha}w_{,\beta}w_{,\beta}\,dU \\ &= -\int_M (2\phi_{,\alpha\beta}\phi_{,\alpha}ww_{,\beta} + \phi_{,\alpha}\phi_{,\alpha}ww_{,\beta\beta})\,dU \\ &= \mathcal{O}\Big(\int_M(\phi_{,\alpha\beta}\phi_{,\alpha\beta} + \phi^{-2}\phi_{,\alpha}\phi_{,\alpha}\phi_{,\beta}\phi_{,\beta})|w|^2dU\Big) \\ &\quad + \sigma\Big(\int_M(\phi_{,\alpha}\phi_{,\alpha}w_{,\beta}w_{,\beta} + |\phi\Delta w|^2)\,dU\Big) \\ &= \mathcal{O}(\lambda^{-4}\|w\|^2) + \sigma\Big(\sum_{\alpha,\beta}\|\phi_{,\alpha}w_{,\beta}\|^2 + \|\phi\Delta w\|^2\Big).\end{aligned}$$

LEMMA

(A2) $$\|\phi w_{,\alpha\beta}\| = \mathcal{O}(\|\phi\Delta w\| + \lambda^{-2}\|w\|).$$

PROOF

$$\sum_{\alpha,\beta} \|\phi w_{,\alpha\beta}\|^2 = \int_M \phi^2 w_{,\alpha\beta} w_{,\alpha\beta} dU$$

$$= \int_M \phi^2 w_{,\alpha\alpha} w_{,\beta\beta} + 2\phi w_{,\beta\beta}\phi_{,\alpha} w_{,\alpha} - 2\phi w_{,\alpha\beta}\phi_{,\beta} w_{,\alpha}) dU$$

$$= \mathcal{O}(\|\phi\Delta w\|^2 + \sum_{\alpha,\beta} \|\phi_{,\alpha} w_{,\beta}\|^2) + \sigma(\sum_{\alpha,\beta} \|\phi w_{,\alpha\beta}\|^2).$$

Using A1 we obtain A2.

LEMMA

(A3) $\|\phi w_{,\alpha}\| = \mathcal{O}((1+k)\lambda^{-1}\|w\|) + \sigma(k^{-1}\lambda\|\phi\Delta w\|),$

where k is an arbitrary positive number.

PROOF

$$\sum_{\alpha} \|\phi w_{,\alpha}\|^2 = \int_M \phi^2 w_{,\alpha} w_{,\alpha} dU$$

$$= -\int_M (\phi^2 w w_{,\alpha\alpha} + 2\phi w_{,\alpha}\phi_{,\alpha} w) dU$$

$$= \mathcal{O}((k^2+1)\lambda^{-2}\|w\|^2) + \sigma(k^{-2}\lambda^2\|\phi\ w\|^2 + \sum_{\alpha} \|\phi w_{,\alpha}\|^2).$$

LEMMA

(A4) $\|\phi w_{,\alpha} w_{,\beta}\| = \mathcal{O}(\overline{w}(\|\phi\Delta w\| + \lambda^{-2}\|w\|)).$

PROOF

$$\sum_{\alpha,\beta} \|\phi w_{,\alpha} w_{,\beta}\|^2 = \int_M \phi^2 w_{,\alpha} w_{,\alpha} w_{,\beta} w_{,\beta} dU$$

$$= -\int_M (\phi^2 w_{,\alpha} w_{,\alpha} w_{,\beta\beta} + 2\phi^2 w_{,\alpha} w_{,\alpha\beta} w_{,\beta} + 2\phi\phi_{,\beta} w_{,\alpha} w_{,\alpha} w_{,\beta}) w dU$$

$$= \mathcal{O}(\sum_{\alpha,\beta} (\|\phi w w_{,\alpha\beta}\|^2 + \|\phi_{,\alpha} w w_{,\beta}\|^2)) + \sigma(\sum_{\alpha,\beta} \|\phi w_{,\alpha} w_{,\beta}\|^2)$$

$$= \mathcal{O}(\overline{w}^2 \sum_{\alpha,\beta} \|\phi w_{,\alpha\beta}\|^2 + \|\phi_{,\alpha}\phi_{,\alpha}\phi^{-1} w^2\|^2) + \sigma(\sum_{\alpha,\beta} \|\phi w_{,\alpha} w_{,\beta}\|^2)$$

$$= \mathcal{O}(\overline{w}^2(\|\phi\Delta w\|^2 + \lambda^{-4}\|w\|^2)) + \sigma(\sum_{\alpha,\beta} \|\phi w_{,\alpha} w_{,\beta}\|^2),$$

where we have used (A0) and (A2).

More complicated to derive, are estimates involving derivatives with respect to U^3. Let $h(U^1,U^2)$ be defined and of class C^2 in the region $U^\alpha U^\alpha < \lambda^2$, and such that

$$\text{(A5)} \qquad \sqrt{h_{,\alpha}h_{,\alpha}} = \mathcal{O}(h\lambda^{-1}), \quad \sqrt{h_{,\alpha\beta}h_{,\alpha\beta}} = \mathcal{O}(h\lambda^{-2}),$$

and let

$$H = \sup_{U^\alpha U^\alpha<\lambda^2} h(U^1,U^2), \quad H^* = \inf_{U^\alpha U^\alpha<\lambda^2} h(U^1,U^2), \quad w_{,3} = \frac{H}{h}\frac{\partial w}{\partial U^3}.$$

LEMMAS

$$\text{(A6)} \qquad \|\phi_{,\alpha}hw_{,3}\| = \mathcal{O}(\lambda^{-1}\|w\|) + \sigma(\lambda^{-1}\|h^2\phi w_{,33}\|),$$

$$\text{(A7)} \qquad \|\phi hw_{,3}\| = \mathcal{O}(\|w\|) + \sigma(\|\phi h^2 w_{,33}\|),$$

$$\text{(A8)} \qquad (\|\phi hw_{,3\alpha}\|, \ \|(\phi hw)_{,3\alpha}\|, \ \|\phi(hw)_{,3\alpha}\|, \ \|\phi hw_{,\alpha 3}\|)$$
$$= \mathcal{O}(\lambda^{-1}\|w\| + \|h\phi w_{,33}\| + H\|\phi\Delta w\|) + \sigma(\lambda\|\phi\Delta w\|)$$

$$\text{(A9)} \qquad \|\phi w_{,3}w_{,3}h\| = \mathcal{O}(\overline{w}(\|\phi w_{,33}h\| + (H^*)^{-1}\|w\|)).$$

PROOFS

In the proofs of all these inequalities we make use of an extension W of the function w suggested for this purpose by L. Nirenberg. The extension $W(U^1,U^2,U^3)$ is defined in the region $M^1 : U^\alpha U^\alpha < \lambda^2$, $|U^3| < 2h$ by

$$W = \begin{cases} 3w(U^1,U^2,2H-U^3) - 2w(U^1,U^2,3H-2U^3) & \text{for } H < U^3 < 2H, \\ w(U) & \text{for } -H < U^3 < H, \\ 3w(U^1,U^2,-2H-U^3) - 2w(U^1,U^2,-3H-2U^3) & \text{for } -2H < U^3 < -H. \end{cases}$$

Clearly W and $W_{,3}$ are continuous and $W_{,33}$ has a jump discontinuity at $U^3 = \pm H$. Moreover

$$\int_{-2H}^{2H} \left| \frac{\partial^k W}{(\partial U^3)^k} \right| dU^3 = \mathcal{O}\left(\int_{-H}^{H} \left| \frac{\partial^k w}{(\partial U^3)^k} \right|^2 dU^3 \right) \quad \text{for} \quad k = 0,1,2.$$

In analogy to ϕ we introduce the function

$$f(U^3) = 1 - \frac{1}{4} \, {\frac{U^3 U^3}{H^2}}^{2}$$

Then following the proof of (A3),

$$\int_{-H}^{H} h^2 |w_{,3}|^2 dU^3 = \int_{-2H}^{2H} h^2 f^2 |W_{,3}|^2 dU^3 = \mathcal{O}((1+k^2) \int_{-2H}^{2H} |W|^2 dU^3)$$

$$+ o(k^{-2} h^4 \int_{-2H}^{2H} f^2 |W_{,33}|^2 dU^3)$$

$$= \mathcal{O}((1+k^2) \int_{-H}^{H} |w|^2 dU^3) + o(k^{-2} h^4 \int_{-H}^{H} |w_{,33}|^2 dU^3.$$

Setting $k = 1$ multiplying by h^2 and integrating over U^1, U^2, establishes (A7). Multiplying by $h^2 \phi_{,\alpha} \phi_{,\alpha}$, putting $k^2 = (\lambda^2 \phi_{,\alpha} \phi_{,\alpha})^{-1}$ and integrating over U^1, U^2, and noting that $\phi_{,\alpha} = \mathcal{O}(\lambda^{-1})$ gives (A6).

To establish (A8) we observe that

$$\|\phi h w_{,3\alpha}\| = \mathcal{O}(\|(\phi f w)_{,3\alpha} h\| + \|\phi_{,\alpha} f w_{,3} h\|$$

$$+ \|\phi f_{,3} w_{,\alpha} h\| + \|\phi_{,\alpha} f_{,3} w h\|)$$

$$= \mathcal{O}(\|h(\phi f w)_{,3\alpha}\| + \|\phi_{,\alpha} w_{,3} h\| + \|\phi w_{,\alpha}\| + \|\phi_{,\alpha} w\|).$$

Now

$$\sum_{\alpha}\|(\phi fw)_{,3\alpha}h\|^2 = \iiint_M (\phi fw)_{,3\alpha}(\phi fw)_{,3\alpha}h^2 dU^1 dU^2 dU^3$$

$$= \iiint_M h^2\left[\frac{\partial}{\partial U^3}H(h^{-1}\phi wf)_{,\alpha}\right]\left[\frac{\partial}{\partial U^3}H(h^{-1}\phi wf)_{,\alpha}\right]dU^1 dU^2 dU^3$$

$$\leq \iiint_{M^1} h^2H^2\left[\frac{\partial^2}{\partial U^3\partial U^\alpha}(h^{-1}\phi Wf)\right]\left[\frac{\partial}{\partial U^3\partial U^\alpha}(h^{-1}\phi Wf)\right]dU^1 dU^2 dU^3$$

$$\leq \left|\iiint_{M^1} h^2H^2\frac{\partial^2}{(\partial U^3)^2}(h^{-1}\phi Wf)\frac{\partial^2}{(\partial U^\alpha)^2}(h^{-1}\phi Wf)dU^1 dU^2 dU^3\right|$$

$$+ \left|\iiint_{M^1} 2hh_{,\alpha}H^2\frac{\partial^2}{\partial U^3\partial U^\alpha}(h^{-1}\phi Wf)\frac{\partial}{\partial U^3}(h^{-1}\phi Wf)dU^1 dU^2 dU^3\right|$$

$$\leq \left|\iiint_{M^1} h^2(h^{-1}\phi Wf)_{,\alpha\alpha}h\ \phi fW_{,33}+2\phi f_{,3}W_{,3}+\phi f_{,33}W\ dU^1 dU^2 dU^3\right|$$

$$+ \left|2\iiint_{M^1}[h(\phi Wf)_{,3\alpha}][h_{,\alpha}(\phi Wf)_{,3}]dU^1 dU^2 dU^3\right|$$

$$\leq \left|\iiint_{M^1} h^2(h^{-1}\phi Wf)_{,\alpha\alpha}h\phi W_{,33}f dU^1 dU^2 dU^3\right|$$

$$+ \left|2\iiint_{M^1} h^2(h^{-1}\phi Wf)_{,\alpha\alpha}][h^2(\phi f_{,3}W_{,3}+\phi f_{,33}W)]dU^1 dU^2 dU^2\right|$$

$$+ \sum_{\alpha}\left(C\|h(\phi fw)_{,3\alpha}\|^2 + C^{-1}\|\phi h_{,\alpha}w_{,3}\|^2\right)$$

$$\leq \mathcal{C}(H^2[\|\phi w_{,\alpha\alpha}\|^2 + \lambda^{-2}\ \|(\phi w)_{,\alpha}\|^2 + \lambda^{-4}\|\phi w\| + \lambda^{-4}(\|w\|^2+\|\phi w\|^2)]$$

$$+ \|h\phi w_{,33}\|^2 + \lambda^{-2}\|\phi hw_{,3}\|^2 + \lambda^{-2}\|\phi w\|^2)$$

$$+ \sigma\left(\sum_{\alpha}(\|h(\phi w)_{,3\alpha}\|^2 + \lambda^2\|(\phi w)_{,\alpha\alpha}\|^2 + \|(\phi w)_{,\alpha}\|^2) + \lambda^{-2}\|\phi w\|^2\right).$$

Where we have used (A0), (A5) and Schwarz's inequality, and C is a constant. Using (A3), (A6) and (A7), we obtain (A8).

Finally, for the proof of (A9) we have

$$\begin{aligned}\int_{-H}^{H} h^2|w_{,3}|^4 dU^3 &= \mathcal{O}\left(\int_{-2H}^{2H} h^2 f^2 |W_{,3}|^4 dU^3\right)\\ &= \mathcal{O}\left(\overline{W}^2 \int_{-2H}^{2H} (f^2h^2|W_{,33}|^2 + h^{-2}|W|^2) dU^3\right)\\ &= \mathcal{O}\left(\overline{w}^2 \int_{-H}^{H} (|hw_{,33}|^2 + h^{-2}|w|^2) dU^3\right),\end{aligned}$$

where we have followed the reasoning of (A4). Multiplying by $(\phi)^2$ and integrating over U^1, U^2 gives (A9).

SOBOLEV'S INEQUALITY FOR A SLAB. For U in M we have

$$\text{(A10)}\ \sqrt{H}\lambda^{-1}|\phi(U)w(U)| = \mathcal{O}\left(N\|\phi\Delta w\| + N\lambda^{-2}\|w\| + \frac{1}{\lambda^2 N}\|h^2\phi w_{,33}\|\right),$$

$$\text{where}\quad N \geq \sqrt{4+(H\lambda^{-1})^2}.$$

PROOF

Assume without restriction $-H \leq U^3 \leq 0$. Let, for $s > 0$, $V^i = U^i + s\xi^i$ where $\xi^i \cdot \xi_i = 1$ and $0 < \xi^3 < H(N\lambda)^{-1}$. Now $|V^3| \leq H$ if $s \leq N\lambda < \frac{H}{\xi^3}$ and $H\lambda^{-1} < 1$. As a function of s, $V^3(0) = U^3(0)$ and

$$|V^3(N\lambda) - V^3(0)| \leq H.$$

But,

$$\begin{aligned}\sqrt{V^\alpha V^\alpha} &\geq \sqrt{(V^\alpha - U^\alpha)(V^\alpha - U^\alpha)} - \sqrt{U^\alpha U^\alpha} \geq s\sqrt{\xi^\alpha \xi^\alpha} - \lambda\\ &\geq N\lambda\sqrt{1-\xi^3\xi^3} - \lambda \geq \lambda(\sqrt{N^2 - H^2\lambda^{-2}} - 1).\end{aligned}$$

Hence if $N^2 \geq 4 + H^2\lambda^{-2}$, then $\sqrt{V^\alpha V^\alpha} \geq \lambda$. Denote by S the value of s such that $V^\alpha(s)V^\alpha(s) = \lambda^2$. Clearly $0 \leq S \leq N\lambda$. Since $\phi, \phi_{,\alpha}$ vanish for $s \geq S$ we can write:

$$\phi^2(U)w^2(U) = \left(\int_0^S s\frac{d^2}{ds^2}\phi(U+s\xi)w(U+s\xi)ds\right)^2$$

$$= \left(\int_0^S \frac{\partial^2}{\partial U^i \partial U^k}(\phi w)\xi^i\xi^k s ds\right)^2$$

$$\leq N\lambda\int_0^S \left|\frac{\partial^2}{\partial U^i \partial U^k}(\phi w)\xi^i\xi^k\right|^2 s^2 ds.$$

Since $\xi^\alpha\xi^\alpha - 1 = \xi^3\xi^3 \leq H^2(N\lambda)^{-2}$, ξ^α are bounded, we have

$$\phi^2(U)w^2(U) = \mathcal{O}\Big(N\lambda\int_0^S\Big(\sum_{\alpha,\beta}|(\phi w)_{,\alpha\beta}|^2$$

$$+ h^2(N\lambda)^{-2}\sum_{\alpha}|(\phi w)_{,\alpha 3}|^2 + h^4(N\lambda)^{-4}|\phi w_{,33}|^2\Big)s^2 ds\Big).$$

Now integrating over all unit vectors ξ such that $0 < \xi^3 < H(N\lambda)^{-1}$ produces:

$$2\pi H(N\lambda)^{-1}\phi^2(U)w^2(U) =$$

$$\mathcal{O}\Big(N\lambda\Big(\sum_{\alpha,\beta}\|(\phi w)_{,\alpha\beta}\|^2 + (N\lambda)^{-2}\sum_{\alpha}\|h(\phi w)_{,\alpha 3}\|^2 + (N\lambda)^{-4}\|h^2\phi w_{,33}\|^2\Big)\Big).$$

From (A1), (A2), (A3), (A5), (A6), (A7), (A8), follows (A10).

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